Visualizing Vector and Tensor Fields in Electromagnetism

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Vector Visualizations

Vector visualizations of the Maxwell Equations are presented.
They are used to motivate the behavior of an electromagnetic plane wave.

Tensor Visualizations

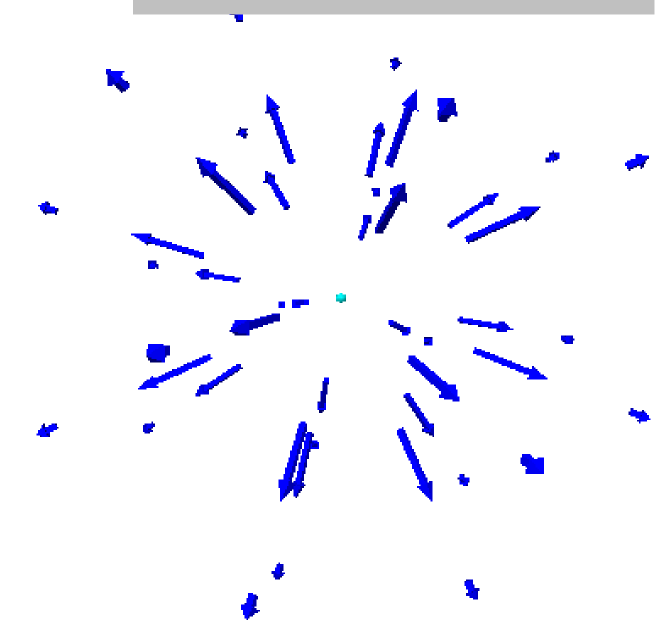
Inspired by Schouten's ***Tensor Calculus for Physicists***, the tensor algebra of vectors and differential forms are accurately represented pictorially. Such visualizations may further develop one's geometric and physical intuition. Applications for electrodynamics and relativity are presented. An early attempt at these visualizations is available at **http://physics.syr.edu/courses/vrml/electromagnetism/** .

New versions are being developed using VPython and will appear at **http://physics.syr.edu/~salgado/software/vpython/** .

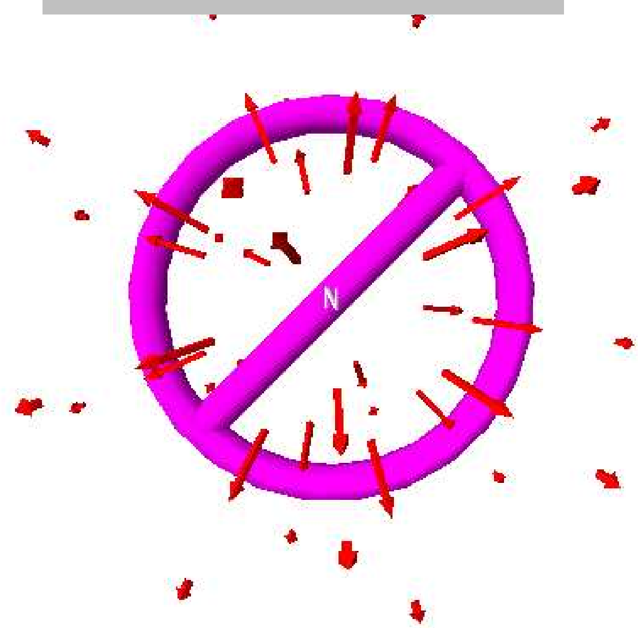
THE MAXWELL EQUATIONS

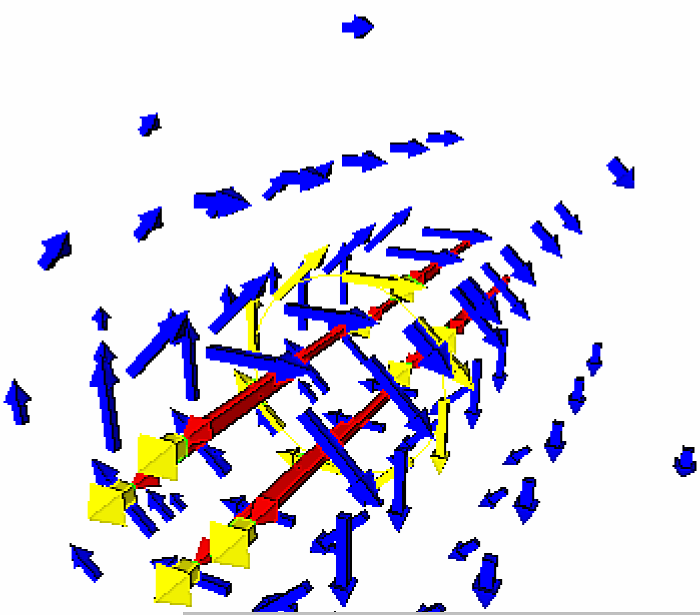
(as vector fields)

Radial E's are associated with electric point charges.

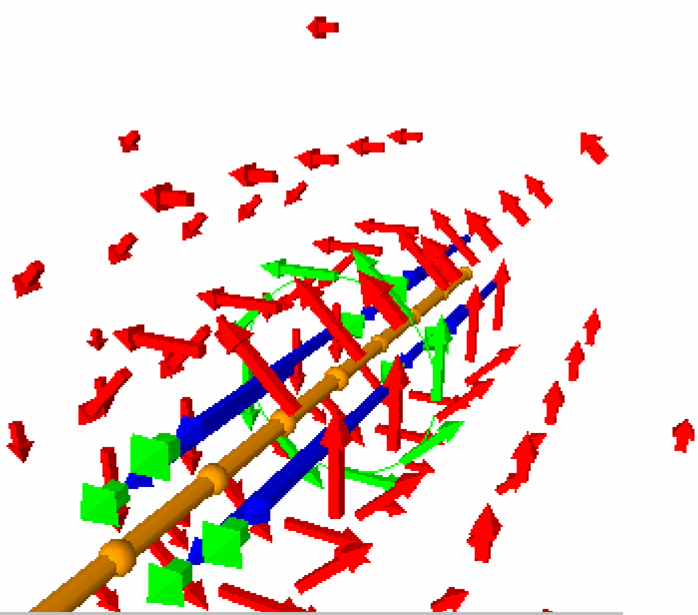


There are no Radial B's.



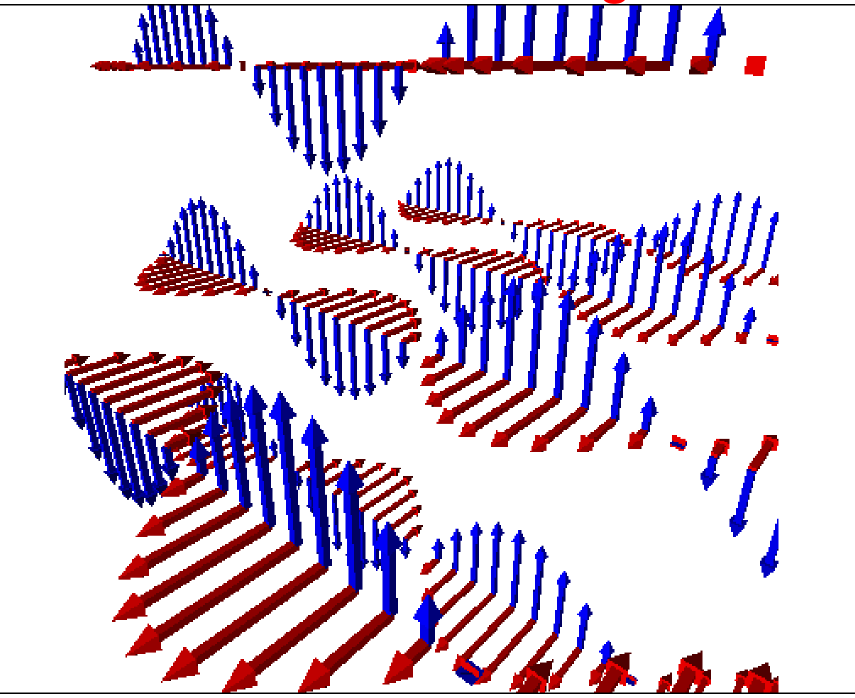


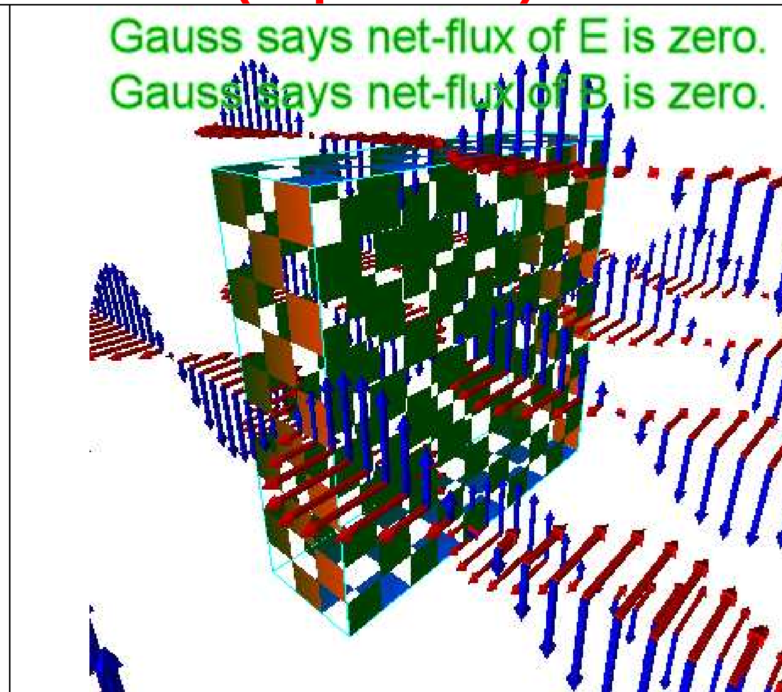
Anti-Curly E's are associated with time-varying B's.

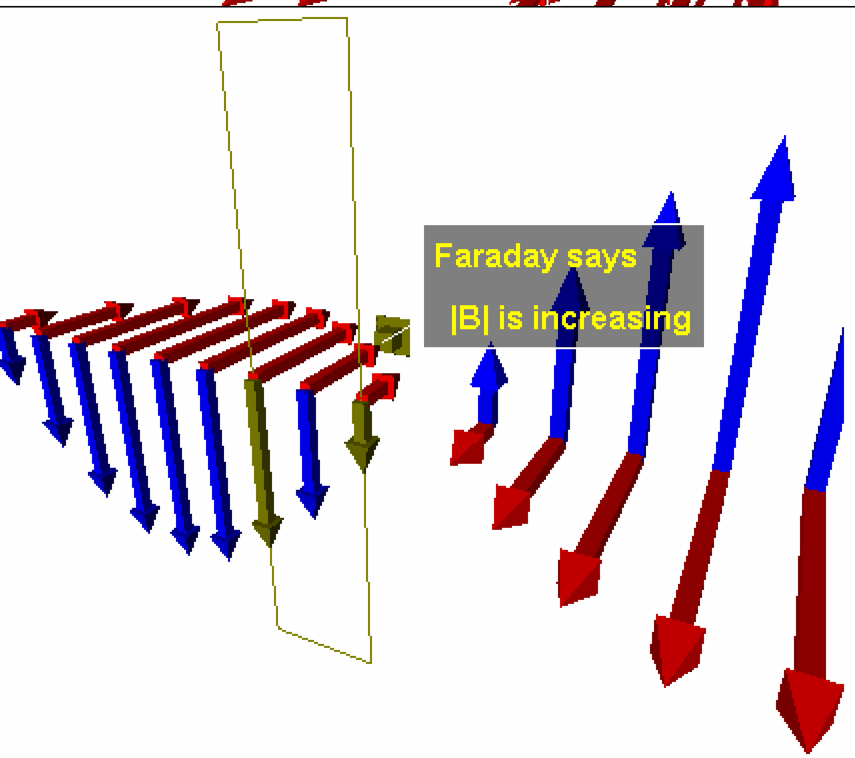


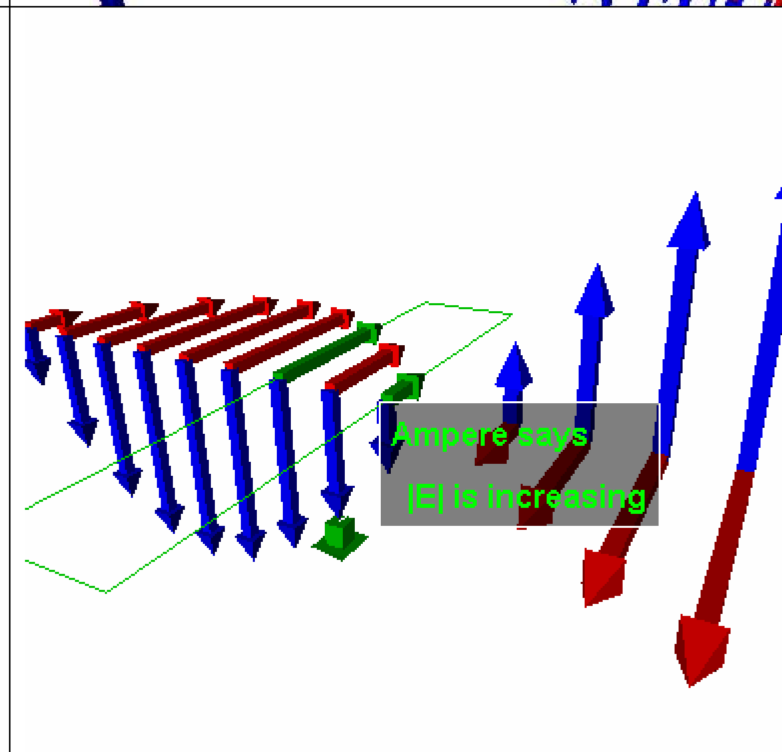
Curly B's are associated with electric currents and time-varying E's.

The Electromagnetic Plane Wave (in pictures)



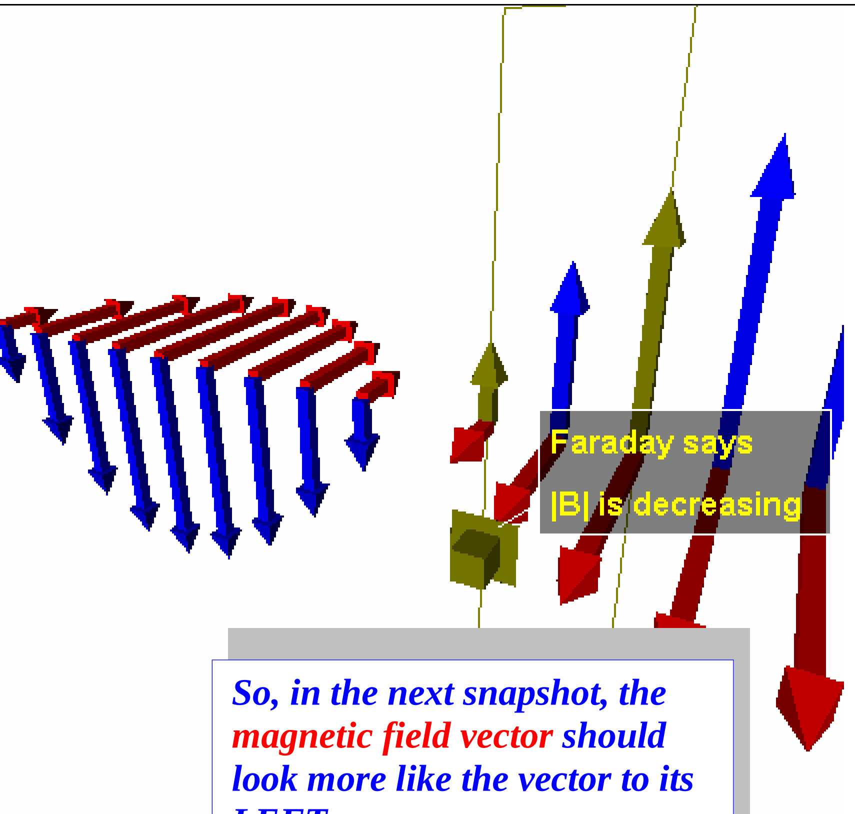




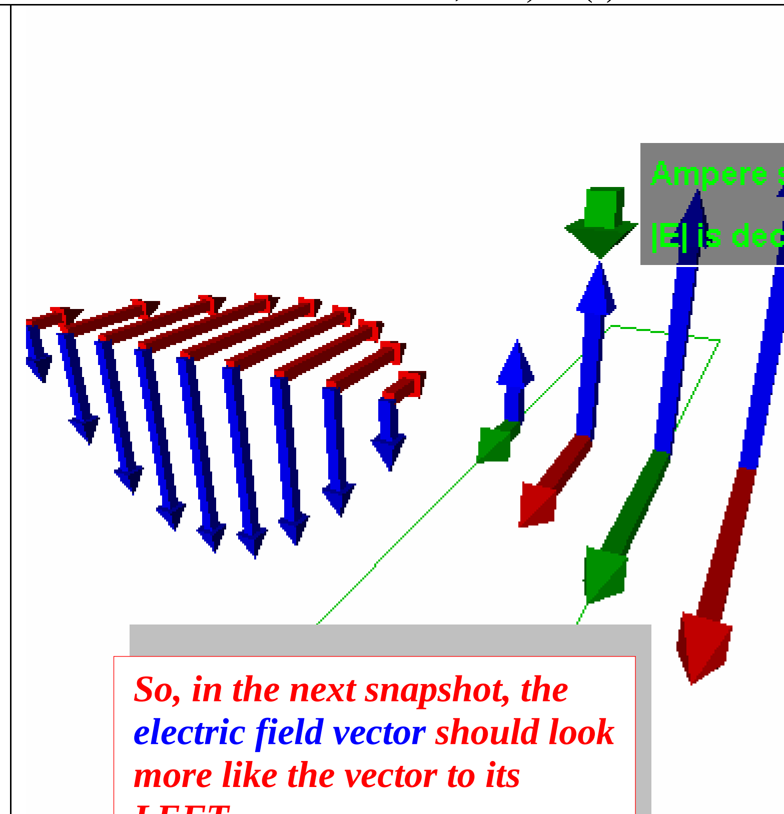


So, in the next snapshot, the magnetic field vector should look more like the vector to its LEFT.

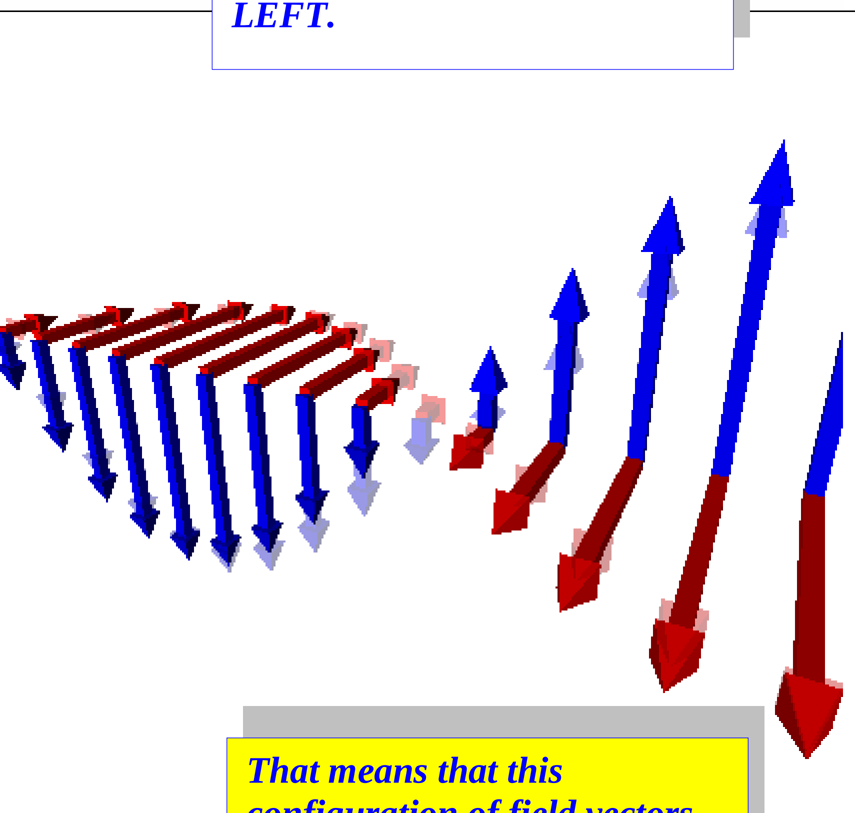
So, in the next snapshot, the electric field vector should look more like the vector to its LEFT.



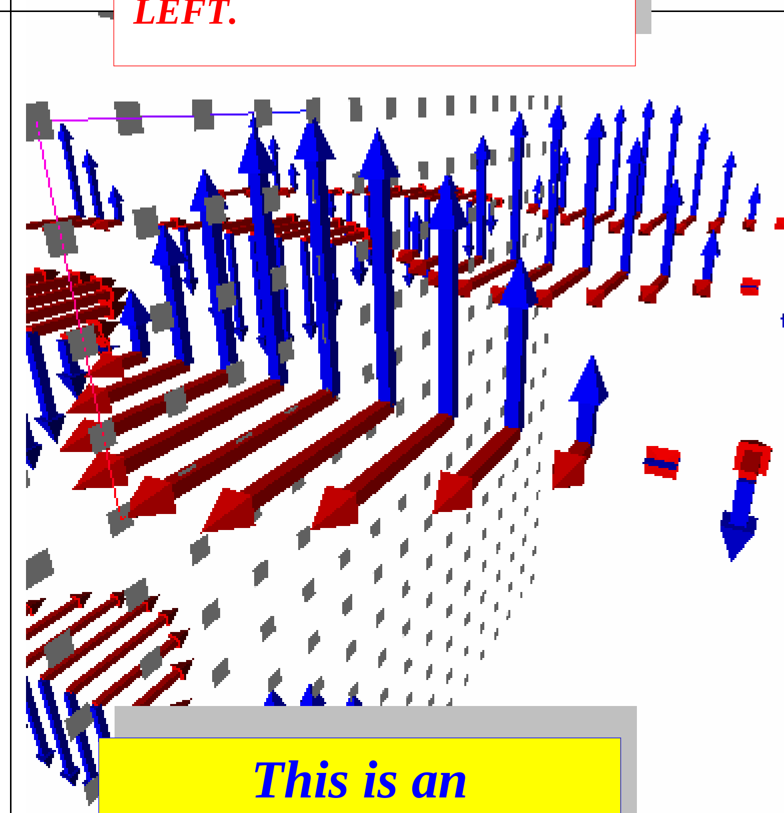
So, in the next snapshot, the magnetic field vector should look more like the vector to its LEFT.



So, in the next snapshot, the electric field vector should look more like the vector to its LEFT.



That means that this configuration of field vectors should slide to the RIGHT.



This is an electromagnetic plane wave.

Note the wavefront and the rectangle representing the Poynting vector.

All vectors are NOT created equal.

The directed quantities

- ***displacements***
- ***gradients***
- ***"normals" to surfaces***
- ***fluxes***

appear to be vectors because of symmetries due to

- ***dimensionality of the vector space***
- ***orientability of the vector space***
- ***existence of a "volume-form"***
- ***existence of a "metric tensor"***
- ***signature of the metric***

These symmetries blur the true nature of the directed quantity.

What is **vector**?

"something with a magnitude and direction"?

Well... no... that's a "Euclidean Vector"
(that is, a vector with a **metric** [a rule for giving the lengths of vectors and the angles between vectors])
Not all vectors in physics are Euclidean vectors.

A **vector space** is a set with the properties of

- addition
  (the sum of two vectors is a vector)
- scalar multiplication
  (the product of a scalar and a vector is a vector)

Elements of this set are called **vectors**.

What is **tensor**?

A **tensor** [of rank n] is a multilinear function of n vectors (that is, inputting n vectors produces a scalar).
They are useful for describing *anisotropic* (direction-dependent) physical quantities.
For example,

- metric tensor
- moment of inertia tensor
- elasticity tensor
- conductivity tensor
- electromagnetic field tensor
- stress tensor
- riemann curvature tensor

If the vector has, for example, 3 components, then a rank-n tensor has $\mathbf{3^n}$ components.

In three dimensions, there are eight directed quantities.

SIMULTANEOUS IDENTIFICATIONS

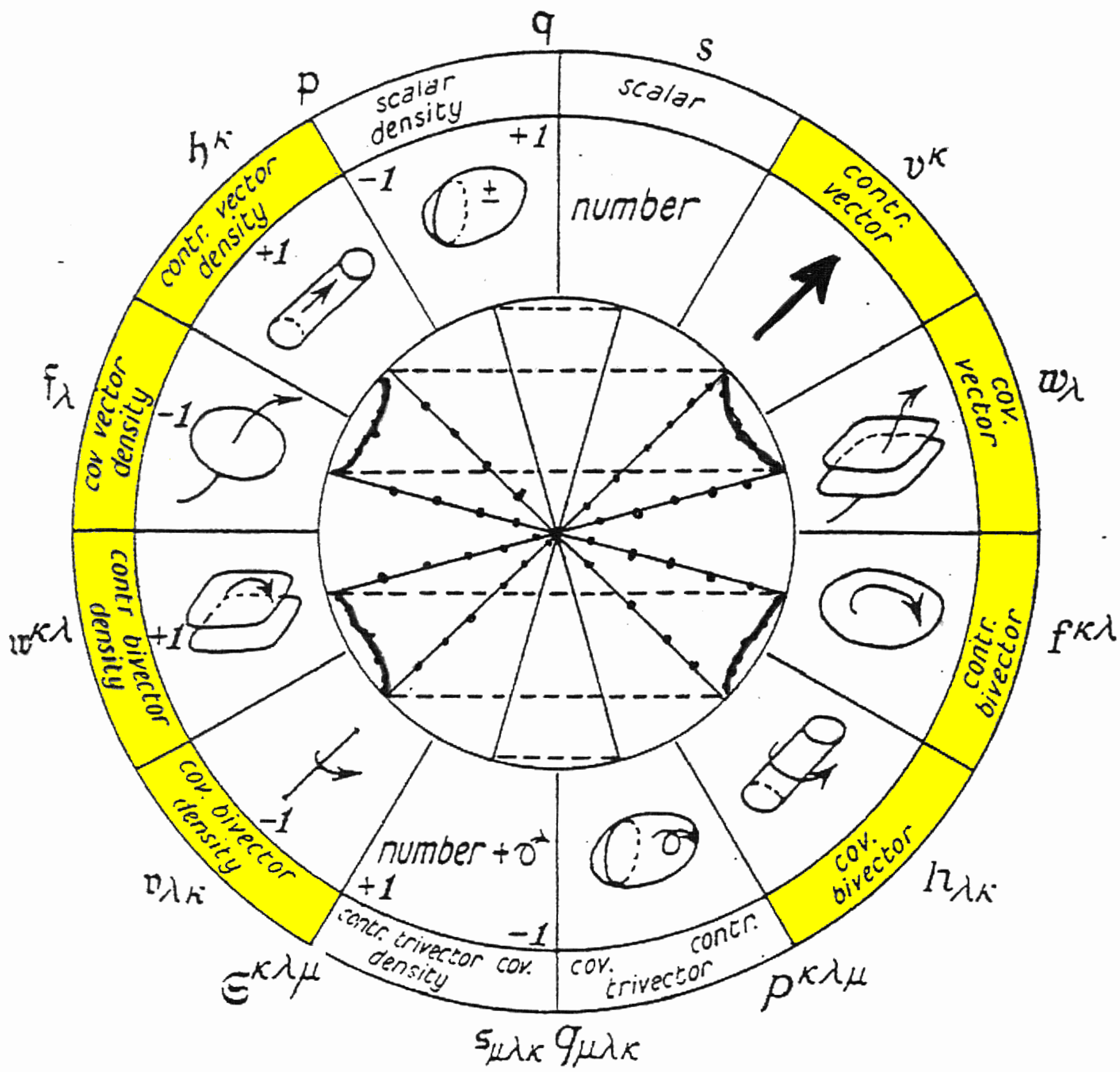


FIG. 13.

From J.A. Schouten, *Tensor Calculus for Physicists*.

VECTORS V^a

Representations

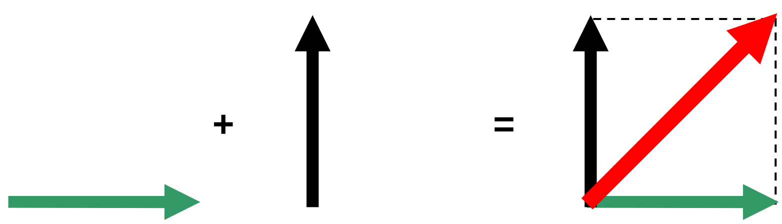
- ordered PAIR OF POINTS with finite separation
- directed line-segment ("an ARROW")

The separation is proportional to its size.

Examples:

- displacement r^a [in meters] as in $U = \frac{1}{2} k_{ab} r^a r^b$
- electric dipole moment $p^a = qd^a$ [in Coulomb-meters] as in $U = -p^a E_a$
- velocity v^a [in meters/sec] as in $K = \frac{1}{2} m_{ab} v^a v^b$

  acceleration a^a [in meters/sec^2] as in $F_a = m_{ab} a^b$



V^a + W^a = $V^a + W^a$

(via the parallelogram rule)

COVECTORS (ONE-FORMS) ω_a

Representations

- ordered PAIR OF PLANES ($\omega_a V^a = 0$ and $\omega_a V^a = 1$) with finite separation
- ("TWIN-BLADES")

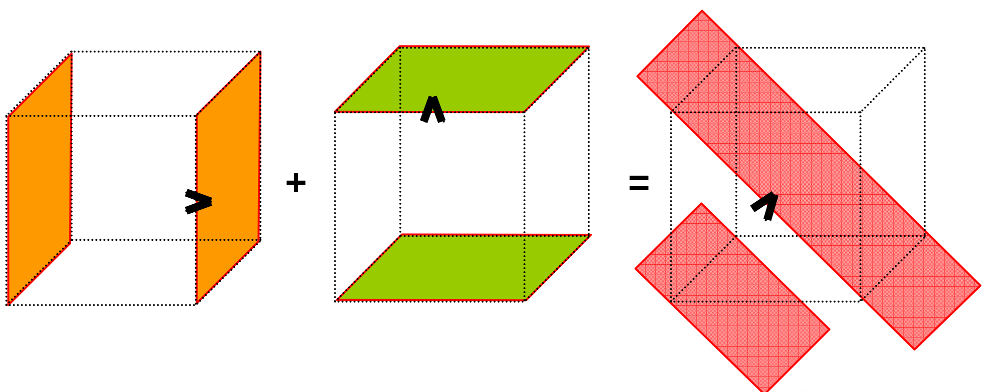
The separation is *inversely*-proportional to its size.

Examples:

- gradient $\nabla_a f$ [in [[f]$\cdot$meters^{-1}]
- conservative force $F_a = -\nabla_a U$ [in Joules/meter] as in $U = -p^a E_a$
- linear momentum "$p_a = \dfrac{\hbar}{\lambda^a}$" [in action/meter]

$$p_a = \frac{\partial S}{\partial q^a} = \frac{\partial L}{\partial \dot{q}^a} \qquad p_a = -\frac{\partial H}{\partial q^a} = F_a$$

- electrostatic field $E_a = -\nabla_a \phi$ [in Volts/meter], $\phi = -\int_\gamma E_a$
- magnetic field $\tilde{H}_a$ [in Amperes/meter] as in $i_{enclosed} = \oint_{\partial A} \tilde{H}_a$



ω_a + η_a = $\omega_a + \eta_a$

(via the co-parallelogram rule)

BIVECTORS A^{ab}

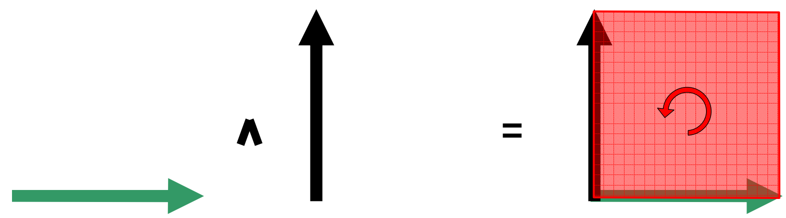
Representations

- ordered PAIR OF VECTORS (via the wedge product)
- directed two-dimensional planar region ("an AREA")

The area is proportional to its size.

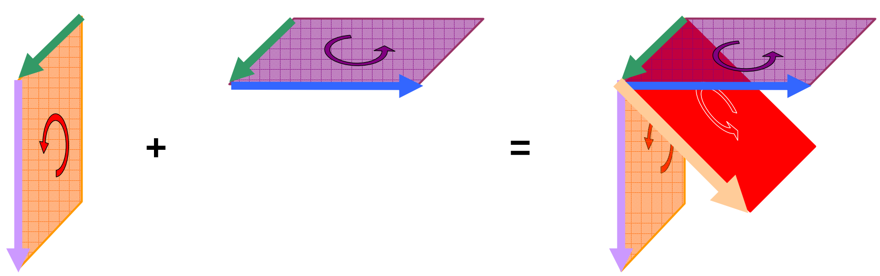
Examples:

- area A^{ab} [in meters^2] as in $A^{ab} = l^{[a}w^{b]}$
- magnetic dipole moment $\mu^{ab} = iA^{ab}$ [in Ampere-meter^2] as in $U = -\mu^{ab}B_{ab}$



$$V^a \wedge W^a = V^{[a}W^{b]}$$

(like the "cross-product")



$$U^{[a}V^{b]} + U^{[a}W^{b]} = U^{[a}(V^{b]} + W^{b]})$$

TWO-FORMS β_{ab}

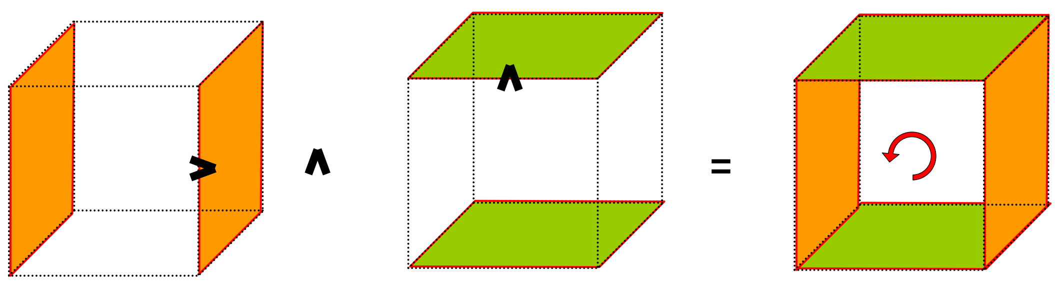
Representations

- ordered PAIR OF CLOSED CURVES
- directed cylinder ("a TUBE") with finite cross-sectional area

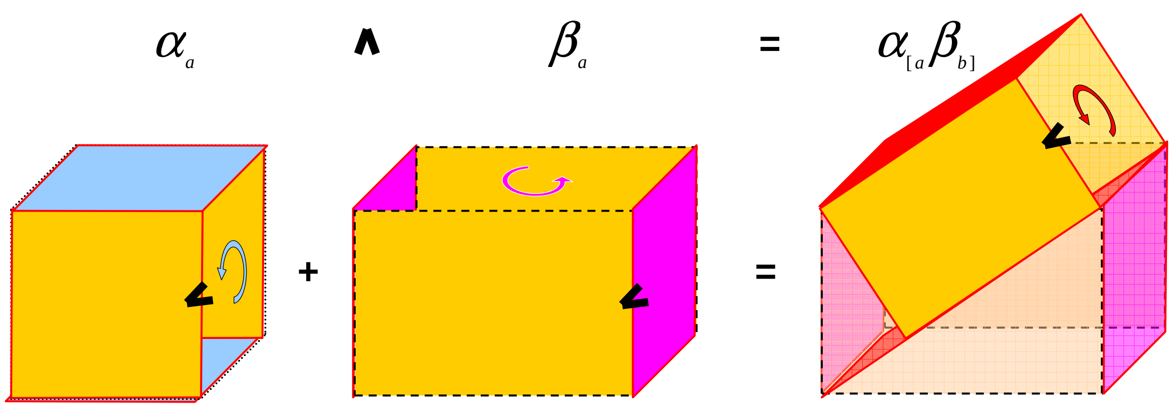
The cross-sectional area is *inversely*-proportional to its size.

Examples:

- magnetic induction B_{ab} [Weber/meter^2=Tesla]
  (magnetic flux per cross-sectional area) as in $\oint\!\!\oint_{\partial V} B_{ab} = 0$
- electric induction $\tilde{D}_{ab}$ [Coulomb/meter^2]
  (electric flux per cross-sectional area) as in $\oint\!\!\oint_{\partial V} \tilde{D}_{ab} = 4\pi q_{enclosed}$
- current density $\tilde{j}_{ab}$ [Ampere/meter^2]
  (charge flux per cross-sectional area) as in $\oint_{\partial A} \tilde{H}_a = \frac{\partial}{\partial t}\iint_A \tilde{D}_{bc} + 4\pi \iint_A \tilde{j}_{bc}$
- Poynting vector $\tilde{S}_{ab} = \frac{1}{4\pi} E_{[a}\tilde{H}_{b]}$ [Watt/meter^2]
  (energy flux per cross-sectional area)

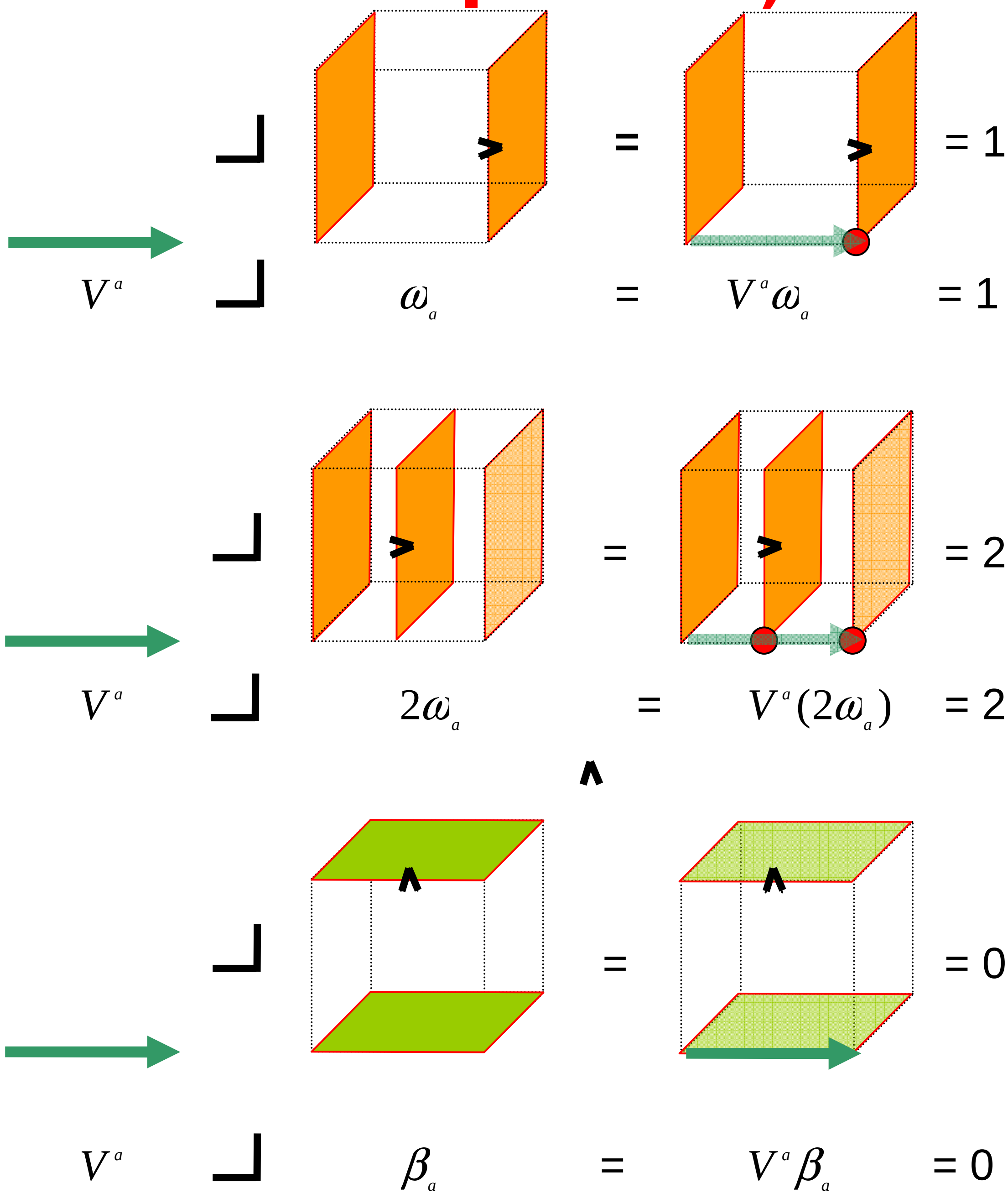


$\alpha_a \wedge \beta_a = \alpha_{[a}\beta_{b]}$



$\alpha_{[a}\beta_{b]} + \alpha_{[a}\gamma_{b]} = \alpha_{[a}(\beta_{b]} + \gamma_{b]})$

TRANSVECTION / INNER PRODUCT (nonmetrical “dot product”) ⌟



In *Gravitation* (Misner, Thorne, Wheeler), this operation is described as counting the “bongs of a bell”.

METRIC TENSOR g_{ab}

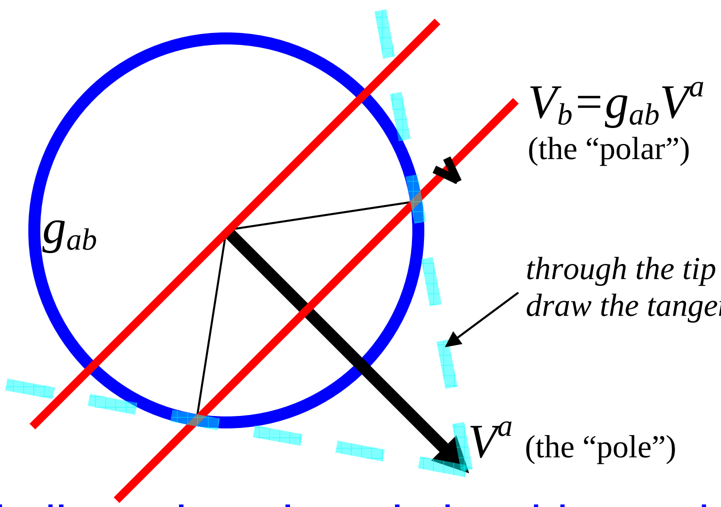
A metric tensor is a symmetric tensor that can be used to assign "magnitudes" to vectors.

$$\|V\|^2 = g_{ab} V^a V^b$$

A metric tensor can also provide a rule to identify a vector with a unique covector. The vector and its covector are "duals" of each other with this metric.

Given a vector V^a, in the presence of a metric, we can form the combination $g_{ab}V^a$, which is a covector denoted by V_b. This is known as "index lowering", a particular move when performing "index gymnastics".

the Euclidean metric:



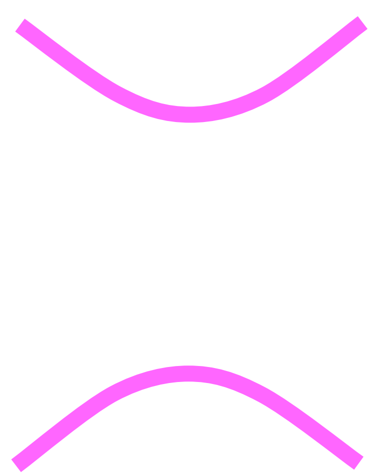
This construction is due to W. Burke, *Applied Differential Geometry*.

See also Burke, *Spacetime, Geometry, and Cosmology*.

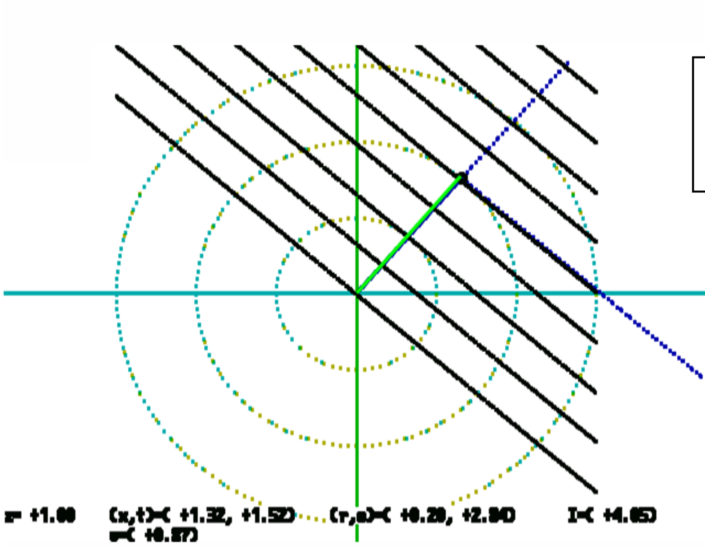
A similar pole-polar relationship can be demonstrated for

Galilean

Minkowski

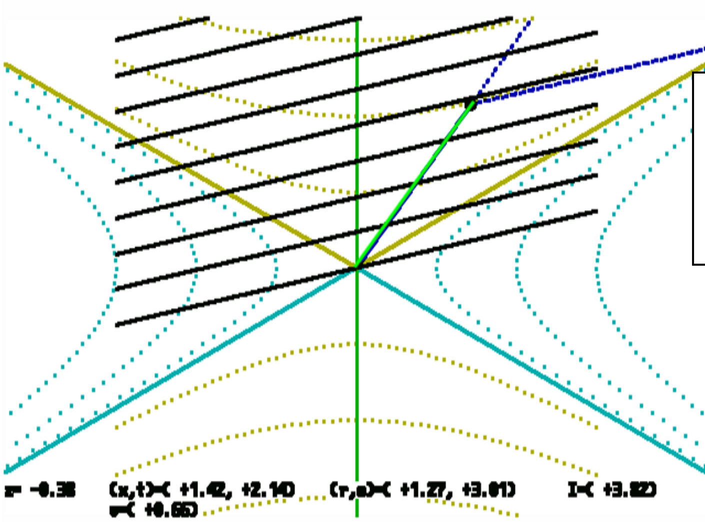


Applications for Relativity

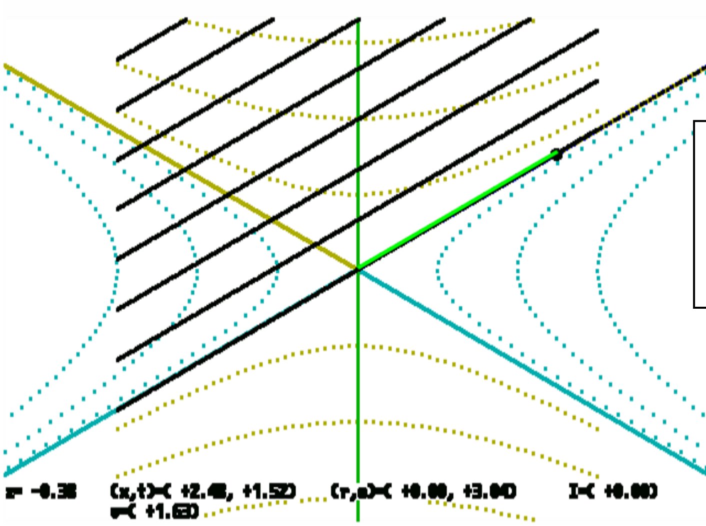


A vector of length 2
with a Euclidean metric.

Note that $V^a(g_{ab}V^b) = (\text{"length of } V^a\text{"})^2$.
Here $V^a(g_{ab}V^b) = 4$.



A timelike vector of
[about] length 2
with a **Minkowski metric.**



A lightlike vector has
zero length with a
Minkowski metric.

In three dimensional space, the following are not directed-quantitities.

TRIVECTORS V^{abc}

Representations

- ordered TRIPLE OF VECTORS
- sensed regions ("a VOLUME") with finite size

The volume is proportional to its size.

Examples:

- volume V^{abc} [in meters^3] as in $V^{abc} = l^{[a}w^b h^{c]}$

THREE-FORMS γ_{abc}

Representations

- ordered TRIPLE OF COVECTORS
- cells ("a BOX") which contain a finite volume

The enclosed-volume is *inversely*-proportional to its size.

Examples:

- charge density $\tilde{\rho}_{abc}$ [in Coulombs/meter^3] as in $q = \iiint_V \tilde{\rho}_{abc}$
- energy density $\tilde{u}_{abc}$ [in Joules/meter^3] as in $\tilde{u}_{abc} = \frac{1}{8\pi} E_{[a}\tilde{D}_{bc]}$

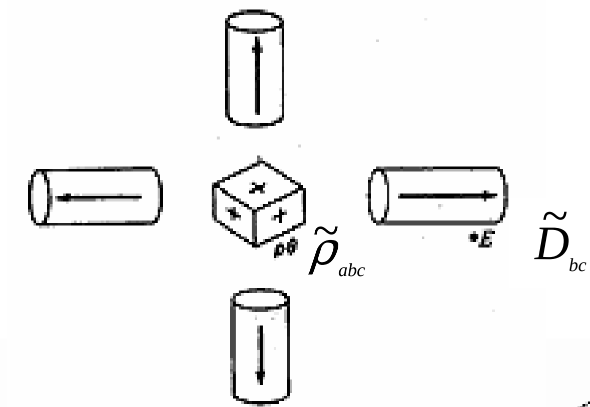
VOLUME FORM ε_{abc}

A volume form provides a rule to identify a vector with a unique two-form (in three dimensions), and vice versa. Vectors that are obtained from [ordinary] two-forms in this way are known as pseudovectors.

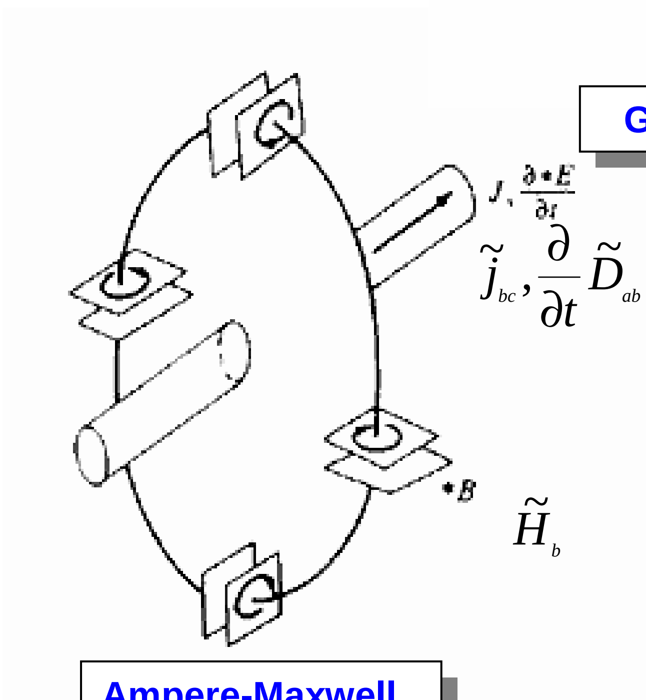
THE MAXWELL EQUATIONS

(as differential forms in Euclidean space)

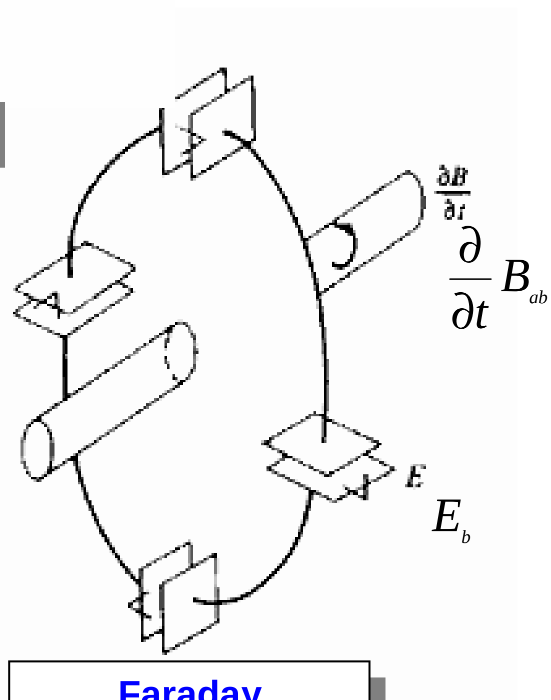
These diagrams are from
W. Burke,
Applied Differential Geometry.



Gauss



Ampere-Maxwell



Faraday

For rendering in 3 dimensions, visit the **VRML Gallery of Electromagnetism** at **physics.syr.edu/courses/vrml/electromagnetism/**
…a new version is being produced using **VPython** at
physics.syr.edu/~salgado/software/vpython/

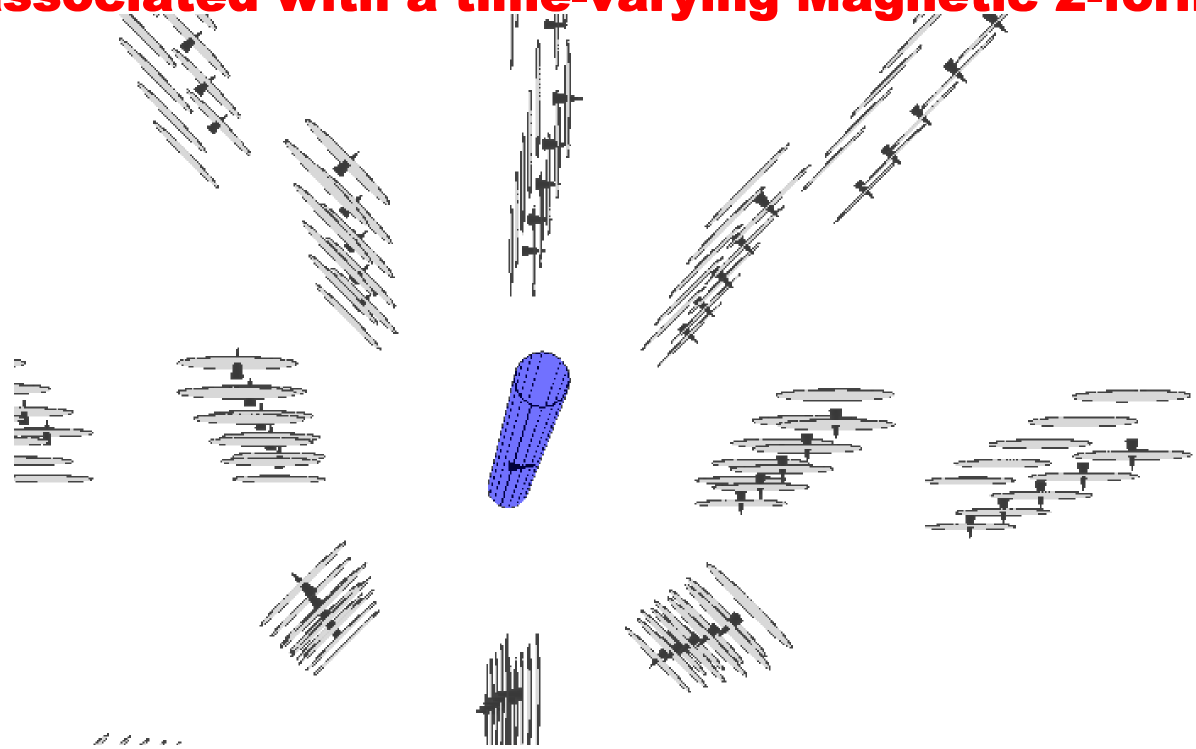
In development…

- ***How do these visualizations transform under a Lorentz boost?***
- ***How do these visualizations arise from the Electromagnetic Field Tensor***

$\mathbf{F_{ab}}$ ***(a differential form in spacetime)?***

The Electric 1-form field of a charged cylinder

Faraday Law: a field of Electric 1-forms associated with a time-varying Magnetic 2-form



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Related online links:

Bill Burke (UC Santa Cruz) **http://ucowww.ucsc.edu/~burke/**
has notes on *"Twisted Forms"* and unfinished draft of "*Div Grad and Curl Are Dead"*

Differential Forms in Electromagnetic Theory **http://www.ee.byu.edu/forms/forms-home.html**
Richard H. Selfridge, David V. Arnold and Karl F. Warnick
(Brigham Young University, Dept of Electrical and Computer Engineering)
"….In the Fall semester of 1995, we completely reworked our beginning electromagnetics course to use differential forms, and developed a set of course notes for use in the class."