Visualizing Tensors

and their Algebra, with applications for Electromagnetism and Relativity

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All vectors are <u>NOT</u> created equal.

The directed quantities

- ***displacements***
- ***gradients***
- ***"normals" to surfaces***
- ***fluxes***

appear to be so due to symmetries

- ***<u>dimensionality</u> of the vector space***
- ***<u>orientability</u> of the vector space***
- ***existence of a "<u>volume-form</u>"***
- ***existence of a "<u>metric tensor</u>"***
- ***<u>signature</u> of the metric***

These symmetries blur the true nature of the directed quantity.

What is vector?

"something with a magnitude and direction"?

Well... no... that's a "Euclidean Vector"
(a vector with a **metric** [a rule for giving the lengths of vectors and the angles between vectors])
Not all vectors in physics are Euclidean vectors.

A **vector space** is a set with the properties of

- addition
  (the sum of two vectors is a vector)
- scalar multiplication
  (the product of a scalar and a vector is a vector)

Elements of this set are called **vectors**.

What is tensor?

A **tensor** [of rank n] is a multi-linear function of n vectors (which, upon inputting n vectors, produces a scalar).

They are useful for describing *anisotropic* (direction-dependent) physical quantities. For example,

- metric tensor
- moment of inertia tensor
- elasticity tensor
- conductivity tensor
- electromagnetic field tensor
- stress tensor
- riemann curvature tensor

If the vector has, for example, 3 components,
then a rank-n tensor has $\mathbf{3^n}$ components.
(If you think about a vector as a column matrix, a tensor can be thought of as a [generalized] matrix. But that's not really a good way to think about them.)

In three dimensions, there are eight types of directed quantities.

SIMULTANEOUS IDENTIFICATIONS

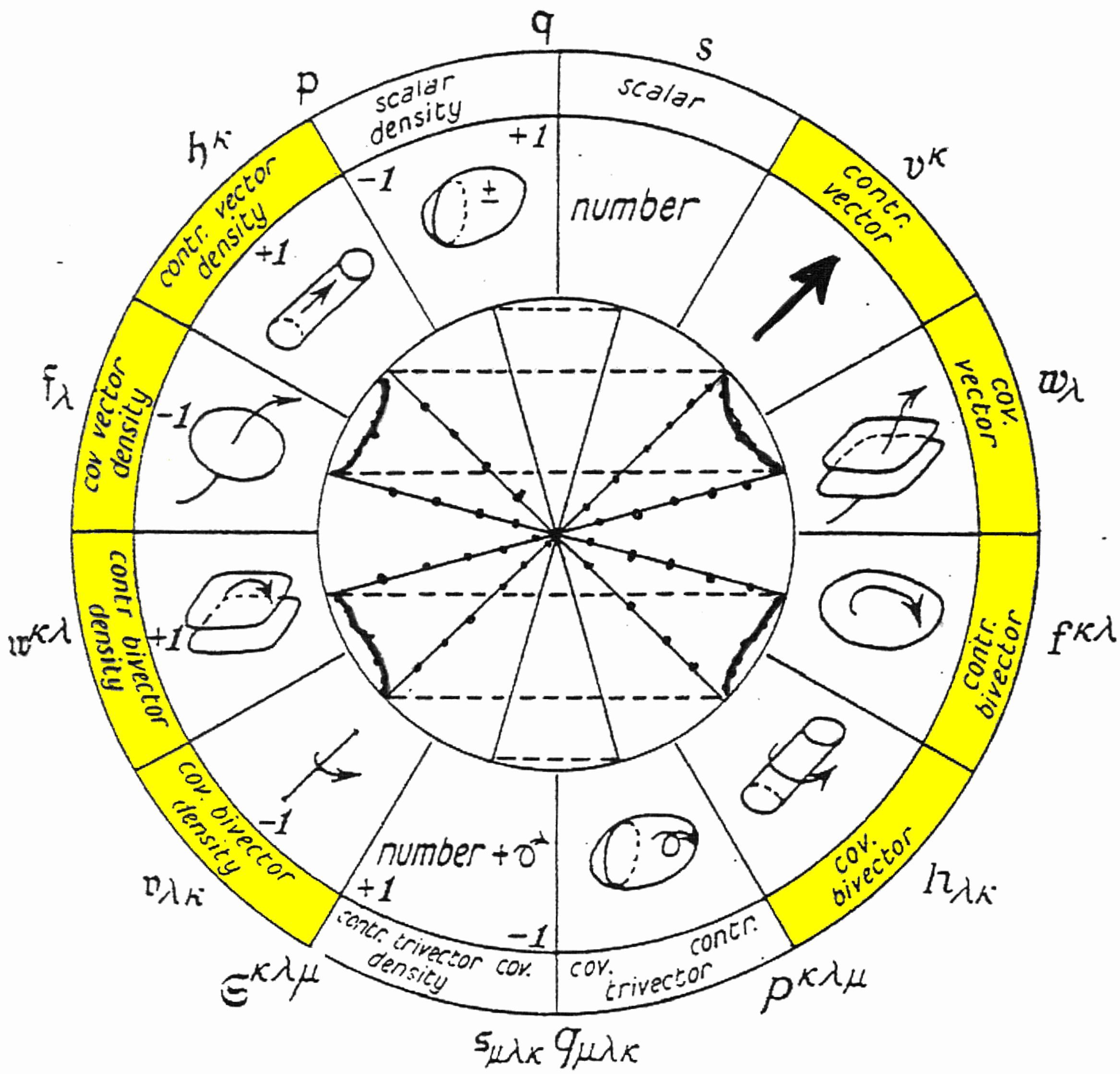


FIG. 13.

From J.A. Schouten, *Tensor Calculus for Physicists*.

A point worth re-emphasizing:
Not all "vectors" in physics were "born as vectors"… they may have been born as covectors (1-forms), bivectors, or 2-forms.

Can we gain some physical and geometrical intuition by visualizing the natural form of these directed-quantities?

VECTORS V^a

a displacement

Representations

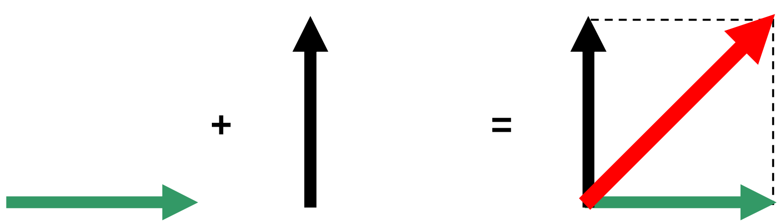
- ordered PAIR OF POINTS with finite separation
- directed line-segment ("an ARROW")

The separation is proportional to its size.

(irrelevant features: thickness of the stem, size of the arrowhead)

Examples:

- displacement r^a [in meters] as in $U = \frac{1}{2} k_{ab} r^a r^b$
- electric dipole moment $p^a = q d^a$ [in Coulomb-meters] as in $U = -p^a E_a$
- velocity v^a [in meters/sec] as in $K = \frac{1}{2} m_{ab} v^a v^b$
  acceleration a^a [in meters/sec^2] as in $F_a = m_{ab} a^b$



V^a + W^a = $V^a + W^a$

(via the parallelogram rule)

COVECTORS (ONE-FORMS) ω_a

a pair of neighboring equipotential surfaces

Representations

- ordered PAIR OF PLANES ($\omega_a V^a = 0$ and $\omega_a V^a = 1$) with finite separation
- ("TWIN-BLADES")

The separation is *inversely*-proportional to its size.

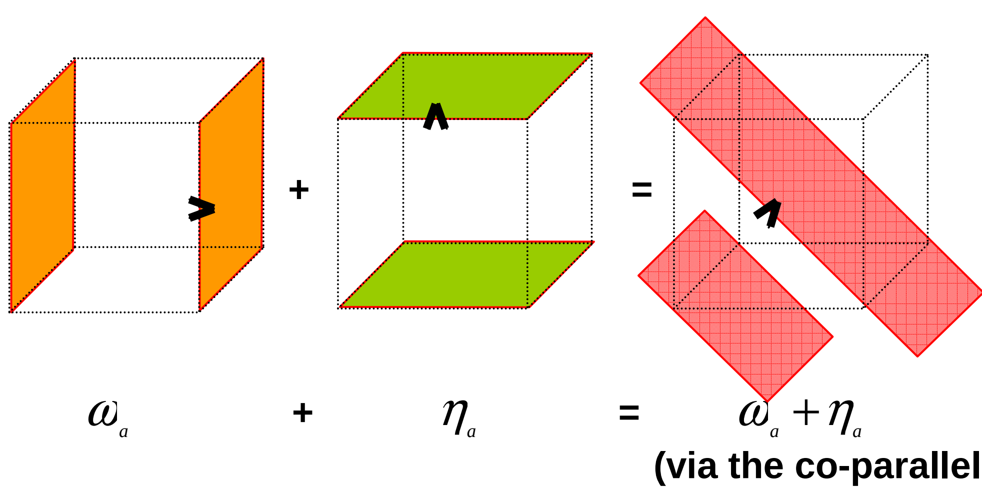
(irrelevant features: size, shape, and alignment of the planar surfaces)

Examples:

- gradient $\nabla_a f$ [in [[f]$\cdot$meters^{-1}]
- conservative force $F_a = -\nabla_a U$ [in Joules/meter] as in $U = -p^a E_a$
- linear momentum "$p_a = \dfrac{\hbar}{\lambda^a}$"$= \hbar k_a$ [in action/meter]

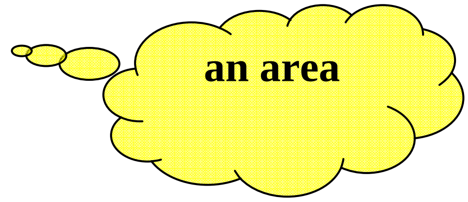
$$p_a = \frac{\partial S}{\partial q^a} = \frac{\partial L}{\partial \dot{q}^a} \qquad p_a = -\frac{\partial H}{\partial q^a} = F_a$$

- electrostatic field $E_a = -\nabla_a \phi$ [in Volts/meter] as in $\phi = -\int_\gamma E_a$
- magnetic field $\tilde{H}_a$ [in Amperes/meter] as in $i_{enclosed} = \oint_{\partial A} \tilde{H}_a$



ω_a + η_a = $\omega_a + \eta_a$

(via the co-parallelogram rule)



BIVECTORS A^{ab}

Representations

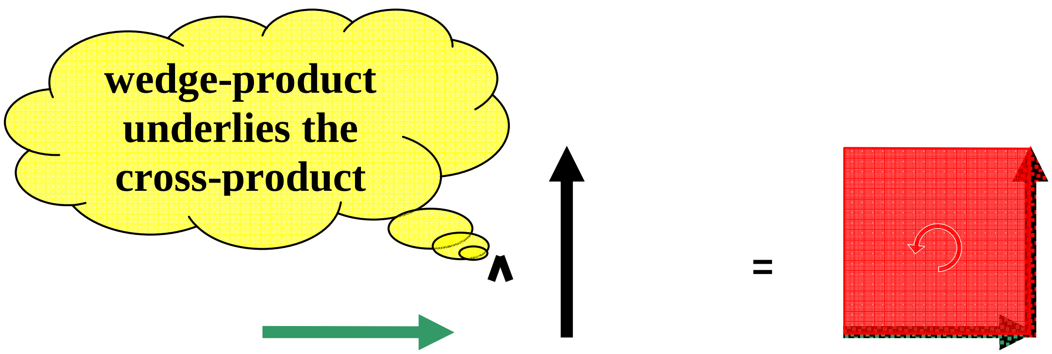
- ordered PAIR OF VECTORS (via the wedge product)
- directed two-dimensional planar region ("an AREA")

The area is proportional to its size.

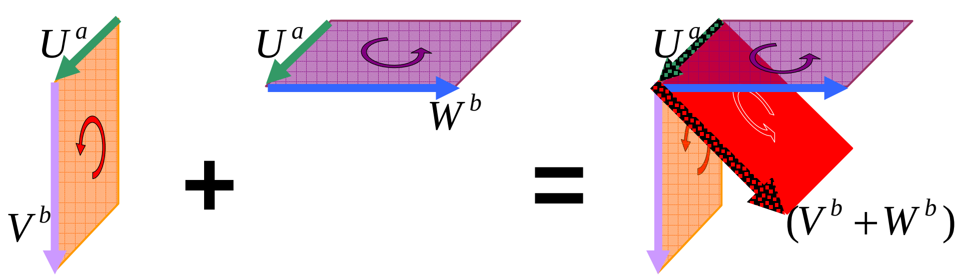
(irrelevant features: shape of the planar surface)

Examples:

- area A^{ab} [in meters²] as in $A^{ab} = l^{[a}w^{b]}$
- force-couple (zero net-force "moment") [in (Newton/meter)-meter²] $M^{ab} = r^{[a}F^{b]}$
- magnetic dipole moment $\mu^{ab} = iA^{ab}$ [in Ampere-meter²] as in $U = -\mu^{ab}B_{ab}$



$$V^a \quad \wedge \quad W^a \quad = \quad V^{[a}W^{b]}$$



$$U^{[a}V^{b]} \quad + \quad U^{[a}W^{b]} \quad = \quad U^{[a}(V^{b]} + W^{b]})$$

TWO-FORMS β_{ab}

a flux-tube:
"density of field lines"

Representations

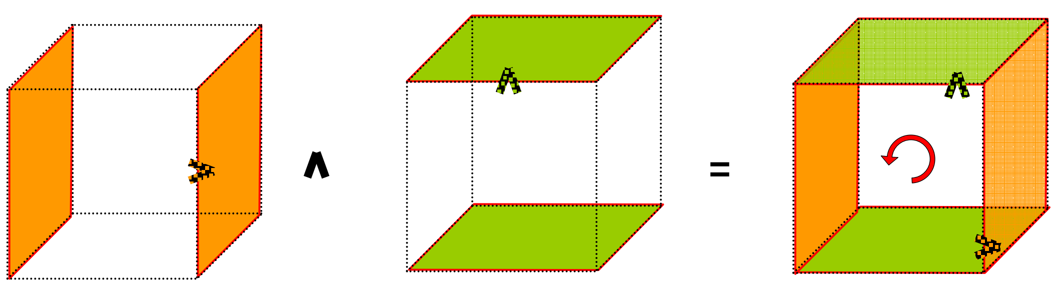
- ordered PAIR OF CLOSED CURVES
- directed cylinder ("a TUBE") with finite cross-sectional area

The cross-sectional area is *inversely*-proportional to its size.

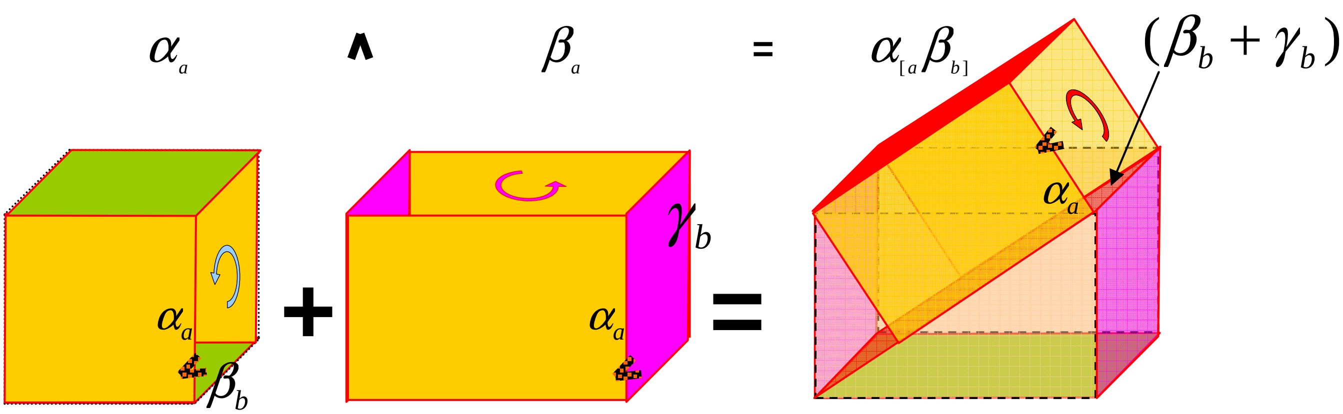
(irrelevant features:
shape of the cross-section, length of the tube)

Examples:

- magnetic induction B_{ab} [Weber/meter^2=Tesla]
  (magnetic flux per cross-sectional area) as in $\oiint_{\partial V} B_{ab} = 0$
- electric induction $\tilde{D}_{ab}$ [Coulomb/meter^2]
  (electric flux per cross-sectional area) as in $\oiint_{\partial V} \tilde{D}_{ab} = 4\pi q_{enclosed}$
- current density $\tilde{j}_{ab}$ [Ampere/meter^2]
  (charge flux per cross-sectional area) as in $\oint_{\partial A} \tilde{H}_a = \frac{\partial}{\partial t}\iint_A \tilde{D}_{bc} + 4\pi \iint_A \tilde{j}_{bc}$
- Poynting vector $\tilde{S}_{ab} = \frac{1}{4\pi} E_{[a} \tilde{H}_{b]}$ [Watt/meter^2]
  (energy flux per cross-sectional area)



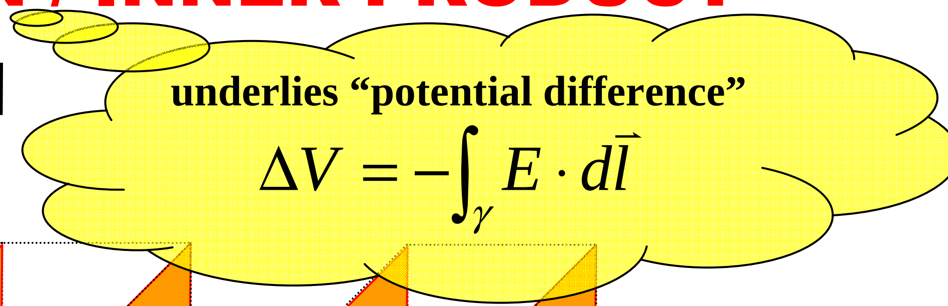
$\alpha_a \quad \wedge \quad \beta_a \quad = \quad \alpha_{[a}\beta_{b]}$

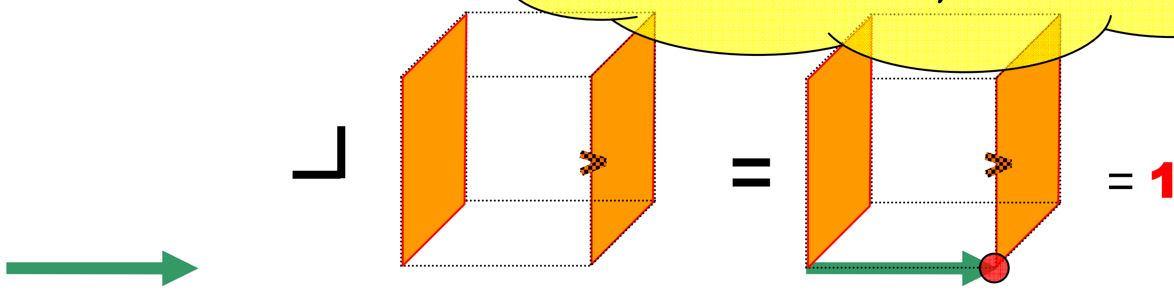


$\alpha_{[a}\beta_{b]} \quad + \quad \alpha_{[a}\gamma_{b]} \quad = \quad \alpha_{[a}(\beta_{b]} + \gamma_{b]})$

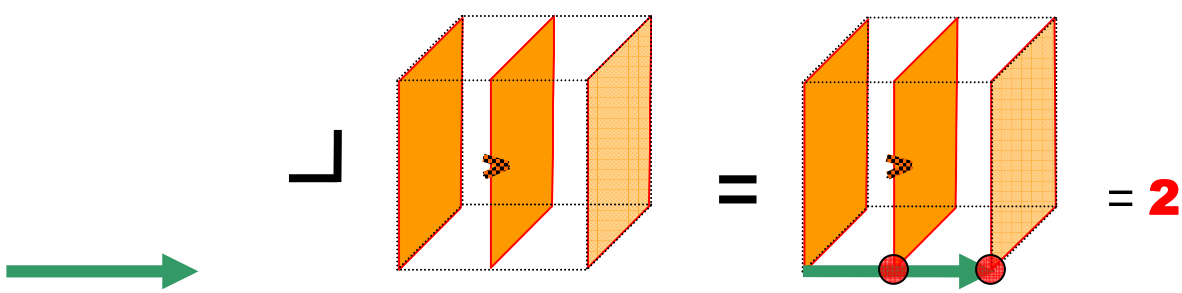
TRANSVECTION / INNER PRODUCT

(nonmetrical ⌟ “dot product”)

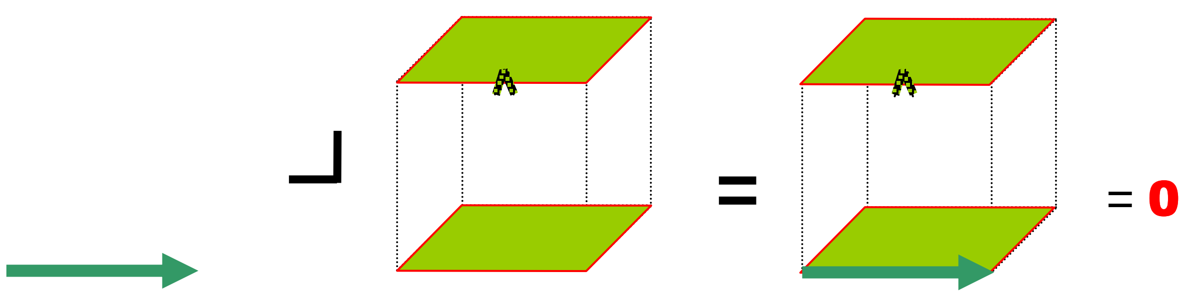




V^a ⌟ ω_a = $V^a\omega_a$ = **1**



V^a ⌟ $2\omega_a$ = $V^a(2\omega_a)$ = **2**



V^a ⌟ β_a = $V^a\beta_a$ = **0**

In *Gravitation* (Misner-Thorne-Wheeler),
this operation is described as counting the “bongs of a bell”.

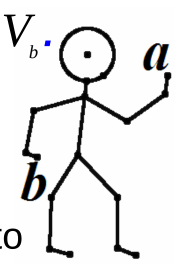
METRIC TENSOR g_{ab}

A metric tensor is a symmetric tensor that can be used to assign "magnitudes" to vectors.

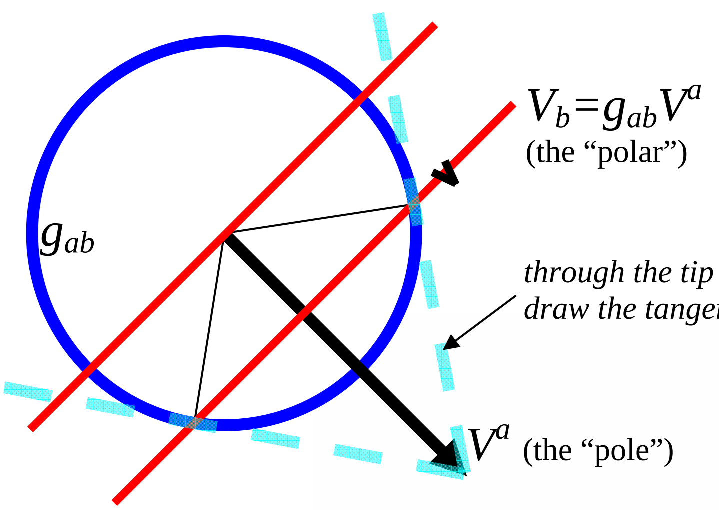
$$\|V\|^2 = g_{ab} V^a V^b$$

A metric tensor can also provide a rule to identify a vector with a unique covector. The vector and its covector are "duals" of each other with this metric.

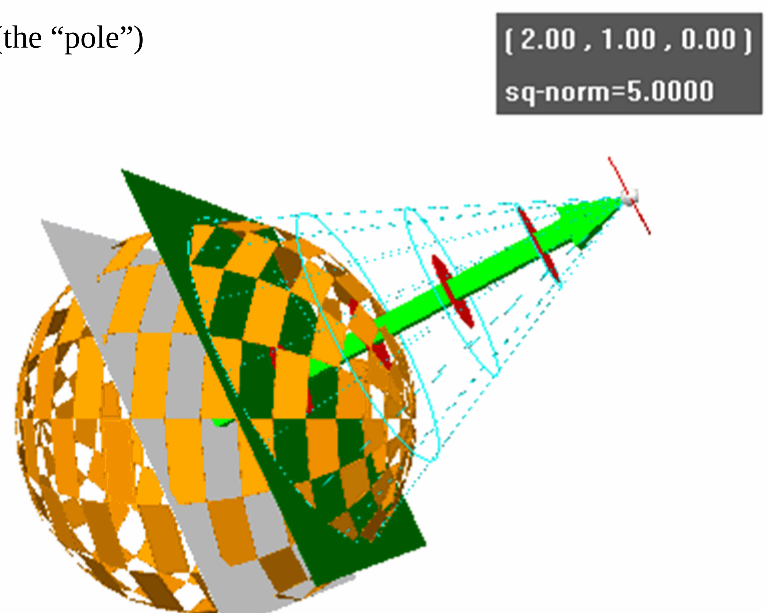
Given a vector V^a, in the presence of a metric,
we can form the combination $g_{ab}V^a$, which is a covector denoted by V_b.
This is known as "index lowering",
a particular move when performing "index gymnastics".



the Euclidean metric:



This construction is due to W. Burke, *Applied Differential Geometry*. (See also Burke, *Spacetime, Geometry, and Cosmology*.)
[First due to Schouten (1923)?]

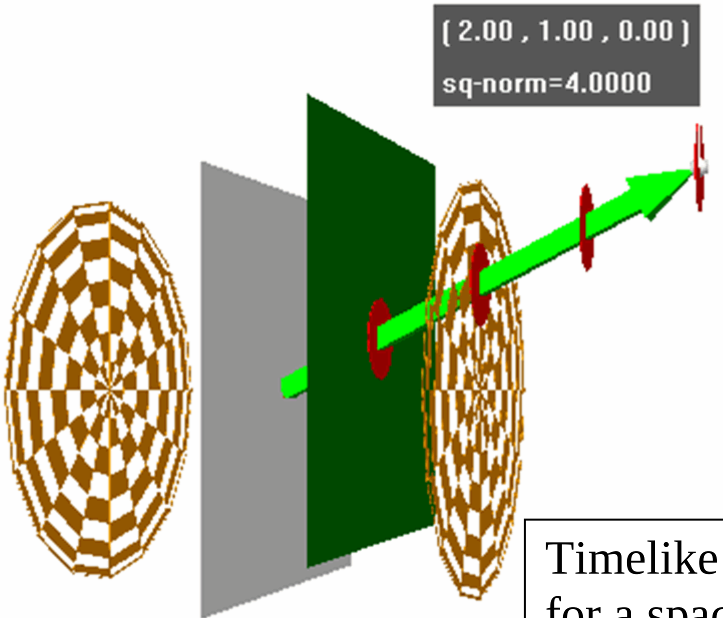


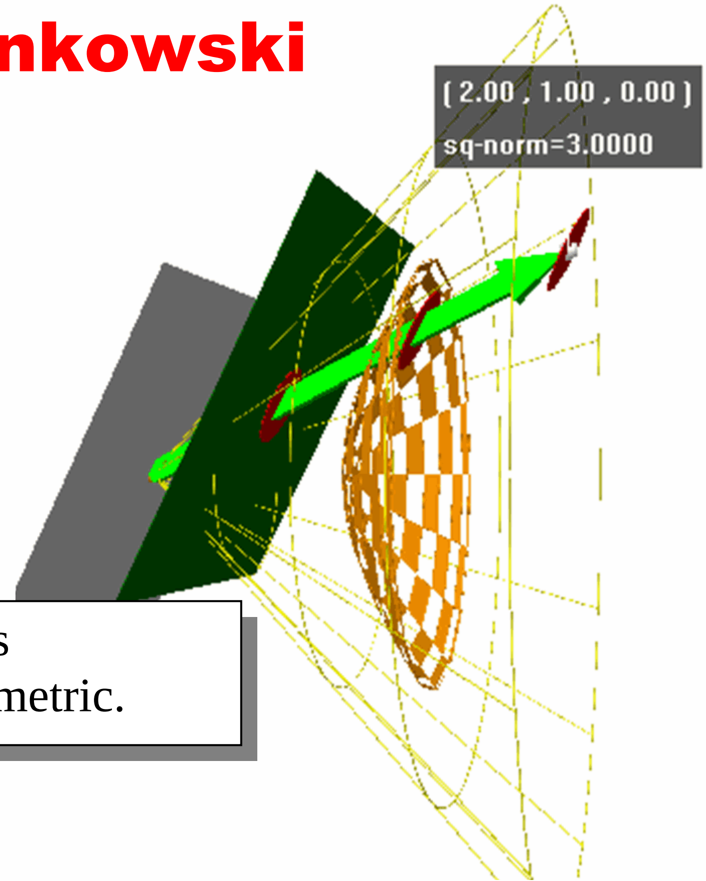
A vector of square-length 5 with a Euclidean metric.

Note that $V^a(g_{ab}V^b) = (\text{"length of } V^a\text{"})^2$.
Here $V^a(g_{ab}V^b) = 5$.

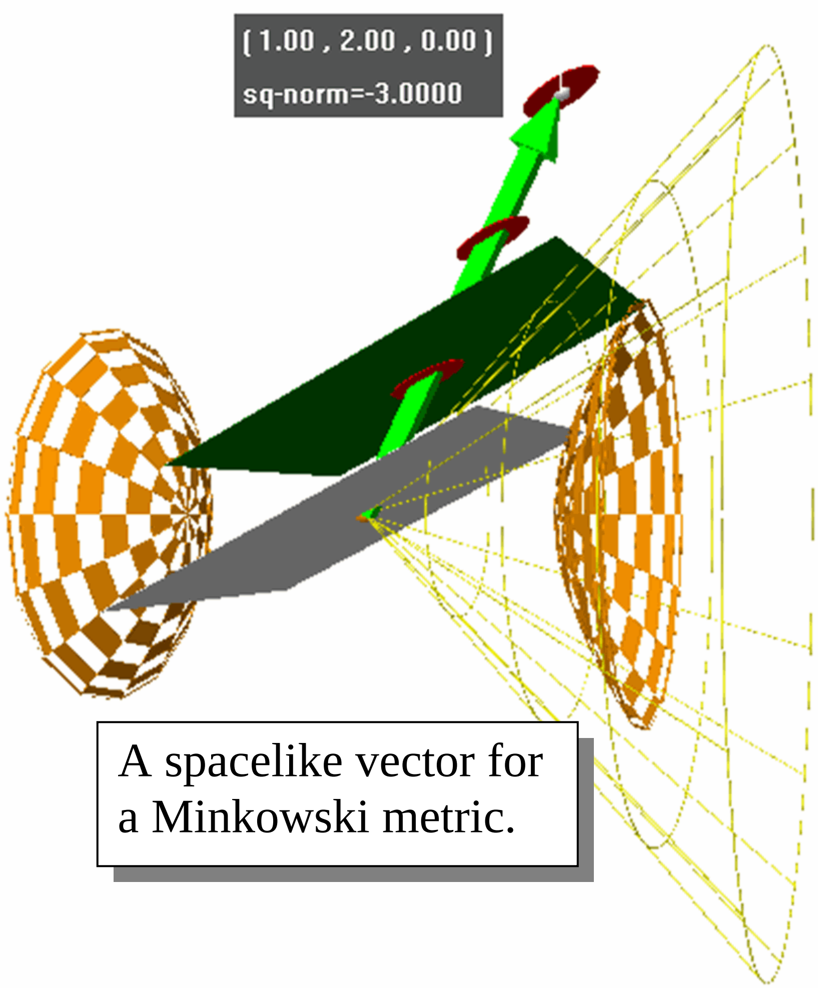
A similar pole-polar relationship can be demonstrated for

Galilean **Minkowski**

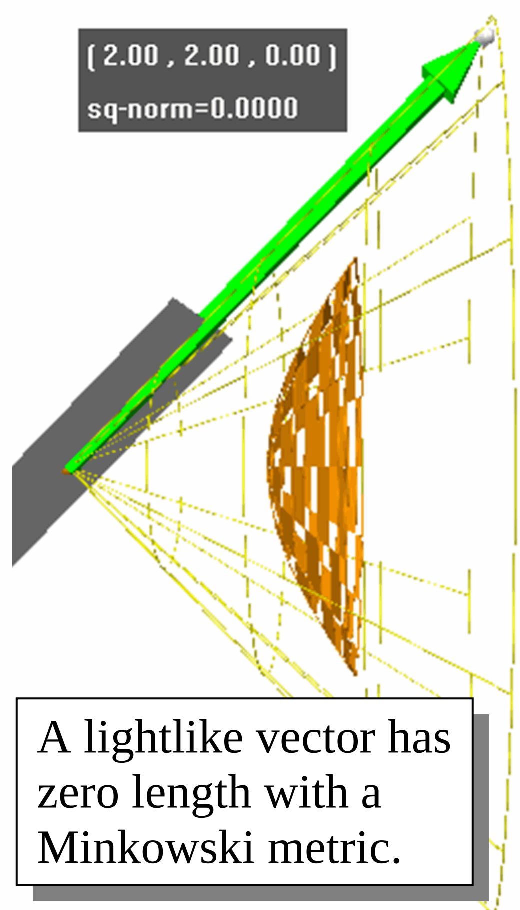




Timelike vectors
for a spacetime metric.

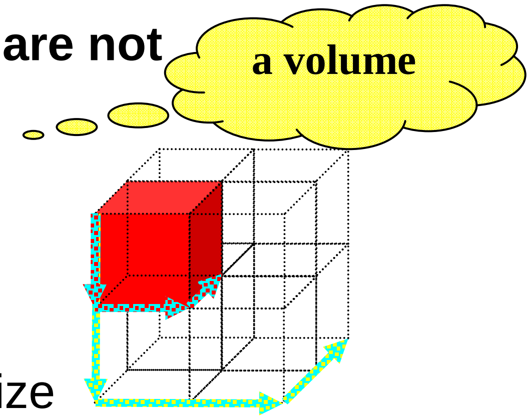


A spacelike vector for
a Minkowski metric.



A lightlike vector has
zero length with a
Minkowski metric.

In three dimensional space, the following are not directed-quantities.



TRIVECTORS V^{abc}

Representations

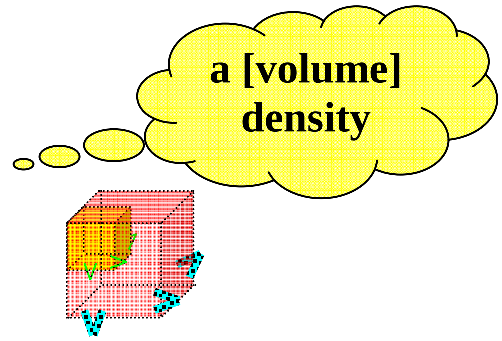
- ordered TRIPLE OF VECTORS
- sensed regions ("a VOLUME") with finite size

The volume is proportional to its size.

(irrelevant features: shape of volume)

Examples:

- volume V^{abc} [in meters^3] as in $V^{abc} = l^{[a} w^{b} h^{c]}$



THREE-FORMS γ_{abc}

Representations

- ordered TRIPLE OF COVECTORS
- cells ("a BOX") which contain a finite volume

The enclosed-volume is *inversely*-proportional to its size.

(irrelevant features: shape of volume)

Examples:

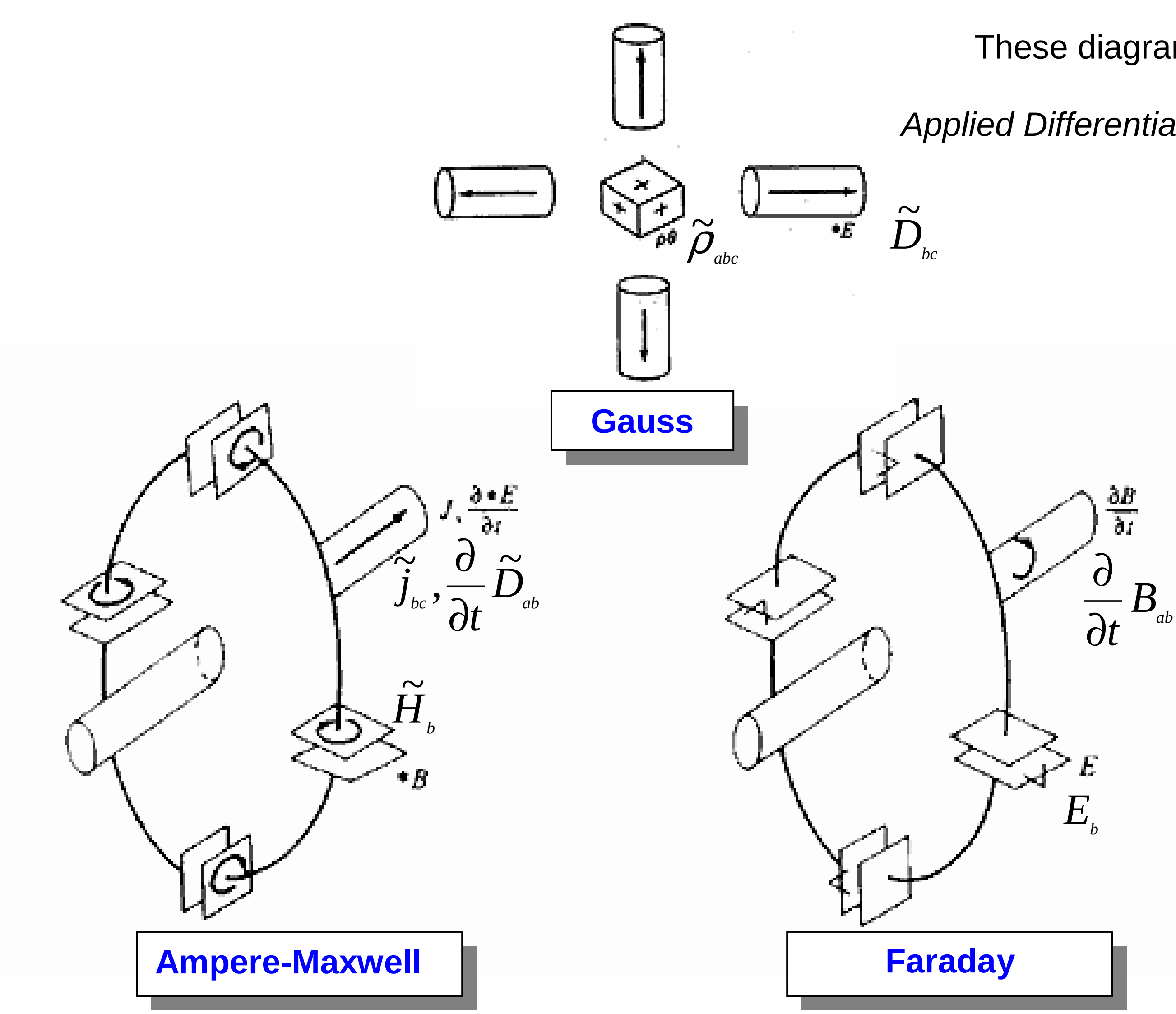
- charge density $\tilde{\rho}_{abc}$ [in Coulombs/meter^3] as in $q = \iiint_V \tilde{\rho}_{abc}$
- energy density $\tilde{u}_{abc}$ [in Joules/meter^3] as in $\tilde{u}_{abc} = \frac{1}{8\pi} E_{[a} \tilde{D}_{bc]}$

VOLUME FORM ε_{abc}

Specifying a volume form provides a rule to identify a vector with a unique two-form (in three dimensions), and vice versa. Vectors that are obtained from [ordinary] two-forms in this way are known as pseudovectors. (Some two-forms can be obtained from bivectors when a metric tensor is specified.)

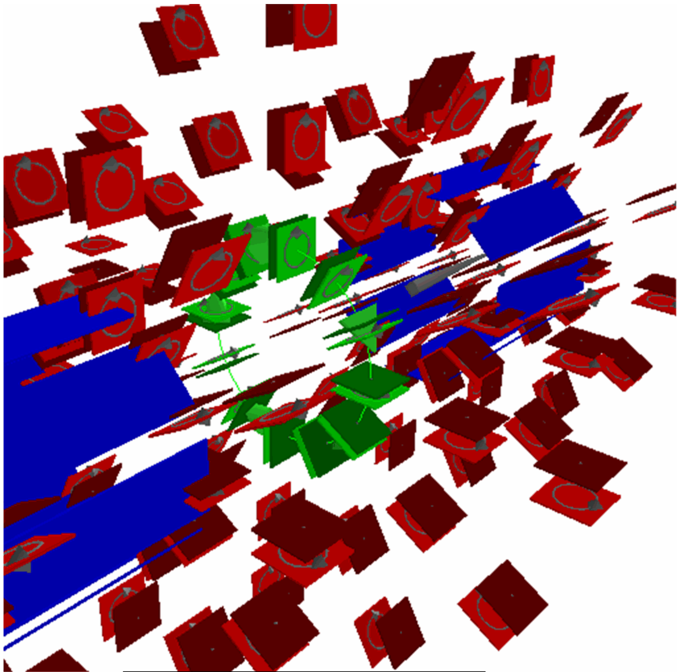
MAXWELL EQUATIONS FOR ELECTROMAGNETISM

These diagrams are from
W. Burke,
Applied Differential Geometry.

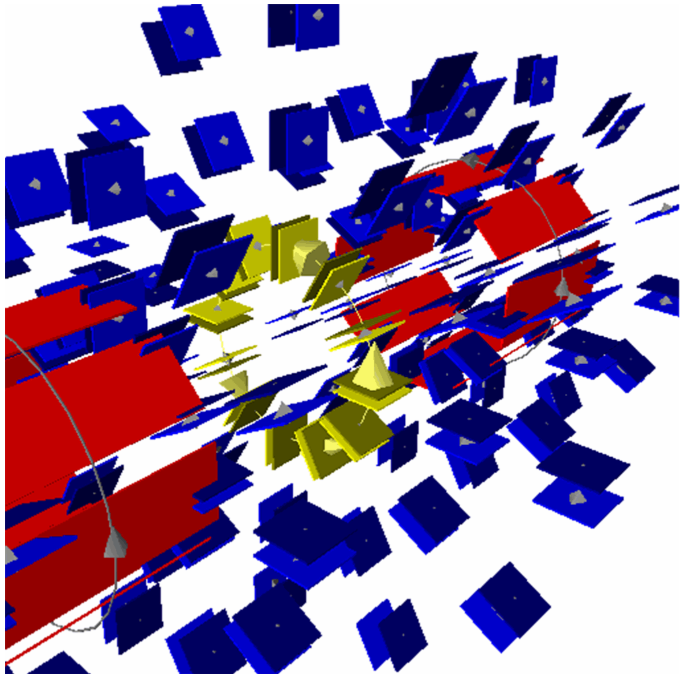


To see these rendered in three dimensions, visit my
VRML Gallery of Electromagnetism (1996)
`physics.syr.edu/courses/vrml/electromagnetism/`

Hopefully soon, it will be available on my VPYTHON page
`physics.syr.edu/~salgado/software/vpython/`



Ampere-Maxwell



Faraday

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