Visual Tensor Calculus

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All vectors are NOT created equal.

The directed quantities

- **displacements**
- **gradients**
- **“normals” to surfaces**
- **fluxes**

appear to be so because of symmetries

- **dimensionality of the vector space**
- **orientability of the vector space**
- **existence of a “volume-form”**
- **existence of a “metric tensor”**
- **signature of the metric**

These symmetries blur the true nature of the directed quantity.

What is vector?

"something with a magnitude and direction"?

Well... no... that's a "Euclidean Vector"
(a vector with a **metric** [a rule for giving the lengths of vectors and the angles between vectors])
Not all vectors in physics are Euclidean vectors.

A **vector space** is a set with the properties of

- addition
  (the sum of two vectors is a vector)
- scalar multiplication
  (the product of a scalar and a vector is a vector)

Elements of this set are called **vectors**.

What is tensor?

A **tensor** [of rank n] is a multilinear function of n vectors (that is, inputting n vectors produces a scalar).
They are useful for describing *anisotropic* (direction-dependent) physical quantities.
For example,

- metric tensor
- moment of inertia tensor
- elasticity tensor
- conductivity tensor
- electromagnetic field tensor
- stress tensor
- riemann curvature tensor

If the vector has, for example, 3 components, then a rank-n tensor has $\mathbf{3}^{n}$ components.

In three dimensions, there are eight directed quantities.

SIMULTANEOUS IDENTIFICATIONS

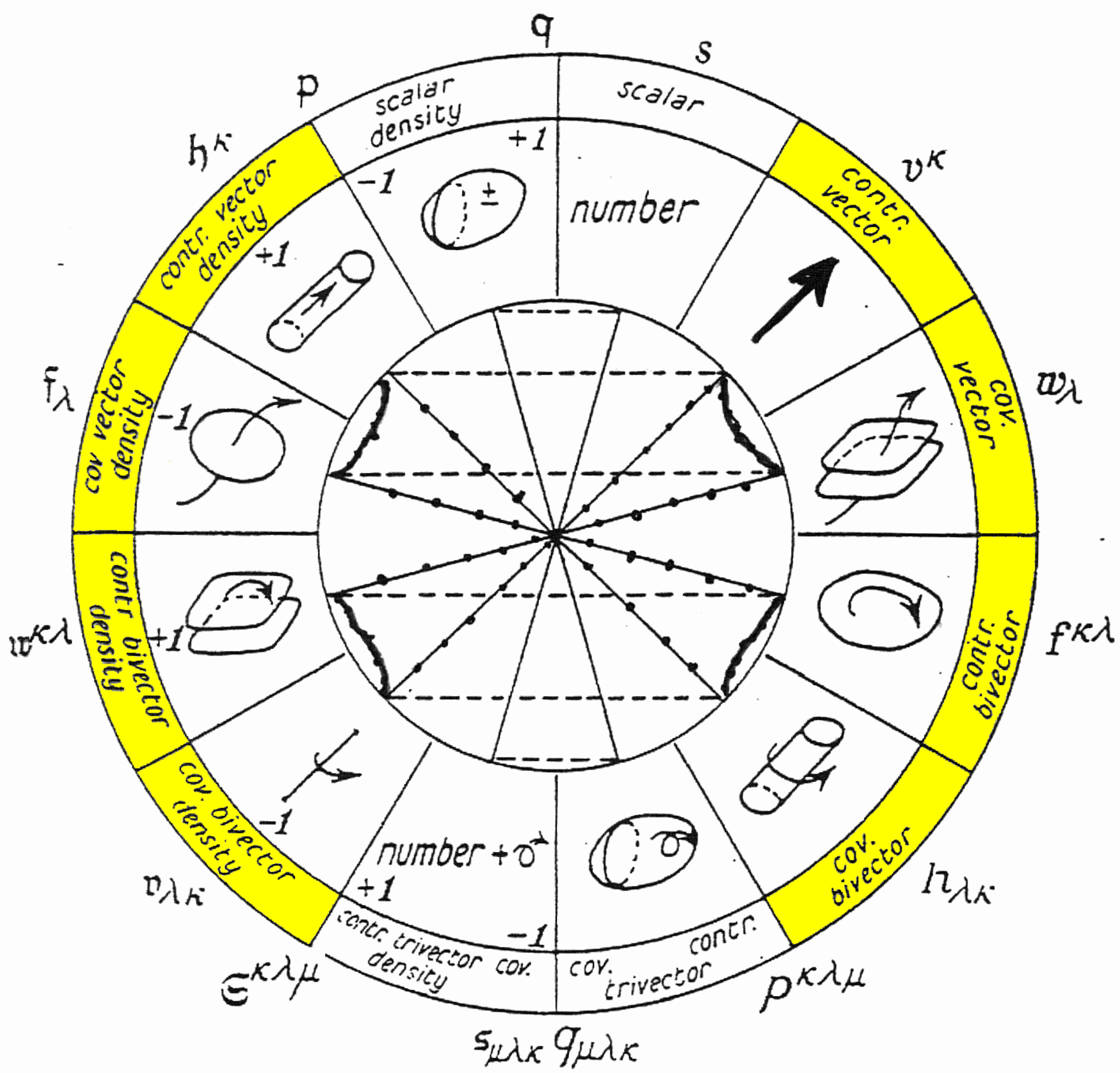


FIG. 13.

From J.A. Schouten, *Tensor Calculus for Physicists*.

VECTORS V^a

Representations

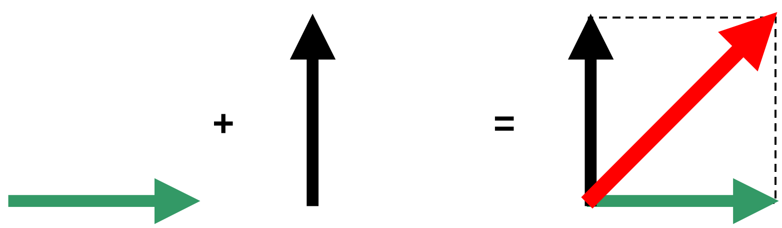
- ordered PAIR OF POINTS with finite separation
- directed line-segment ("an ARROW")

The separation is proportional to its size.

Examples:

- displacement r^a [in meters] as in $U = \frac{1}{2} k_{ab} r^a r^b$
- electric dipole moment $p^a = qd^a$ [in Coulomb-meters] as in $U = -p^a E_a$
- velocity v^a [in meters/sec] as in $K = \frac{1}{2} m_{ab} v^a v^b$

  acceleration a^a [in meters/sec^2] as in $F_a = m_{ab} a^b$



V^a + W^a = $V^a + W^a$

(via the parallelogram rule)

COVECTORS (ONE-FORMS) ω_a

Representations

- ordered PAIR OF PLANES ($\omega_a V^a = 0$ and $\omega_a V^a = 1$) with finite separation
- ("TWIN-BLADES")

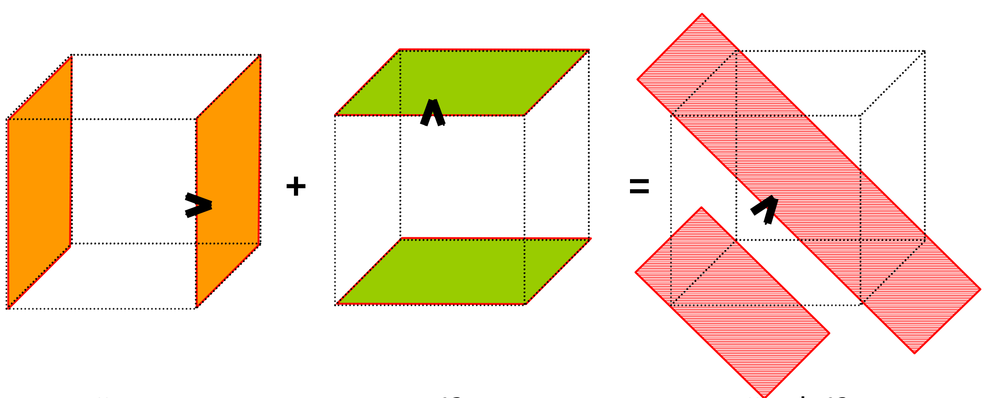
The separation is *inversely*-proportional to its size.

Examples:

- gradient $\nabla_a f$ [in [[f]·meters^{-1}]
- conservative force $F_a = -\nabla_a U$ [in Joules/meter] as in $U = -p^a E_a$
- linear momentum "$p_a = \dfrac{\hbar}{\lambda^a}$" [in action/meter]

$$p_a = \frac{\partial S}{\partial q^a} = \frac{\partial L}{\partial \dot{q}^a} \qquad p_a = -\frac{\partial H}{\partial q^a} = F_a$$

- electrostatic field $E_a = -\nabla_a \phi$ [in Volts/meter], $\phi = -\int_\gamma E_a$
- magnetic field $\tilde{H}_a$ [in Amperes/meter] as in $i_{enclosed} = \oint_{\partial A} \tilde{H}_a$



ω_a + η_a = $\omega_a + \eta_a$

(via the co-parallelogram rule)

BIVECTORS A^{ab}

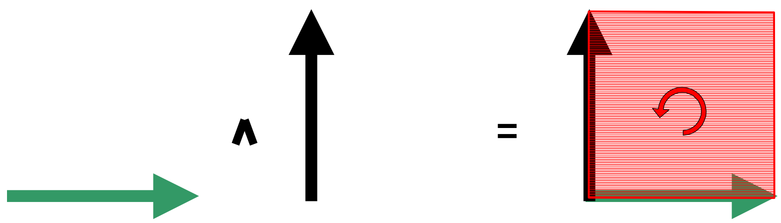
Representations

- ordered PAIR OF VECTORS (via the wedge product)
- directed two-dimensional planar region ("an AREA")

The area is proportional to its size.

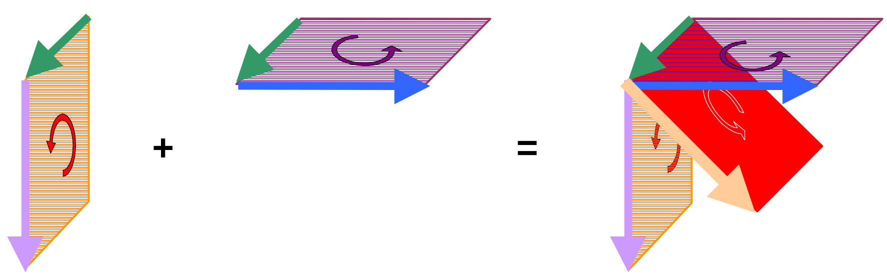
Examples:

- area A^{ab} [in meters^2] as in $A^{ab} = l^{[a}w^{b]}$
- magnetic dipole moment $\mu^{ab} = iA^{ab}$ [in Ampere-meter^2] as in $U = -\mu^{ab}B_{ab}$



$V^a \wedge W^a = V^{[a}W^{b]}$

(like the "cross-product")



$U^{[a}V^{b]} + U^{[a}W^{b]} = U^{[a}(V^{b]} + W^{b]})$

TWO-FORMS β_{ab}

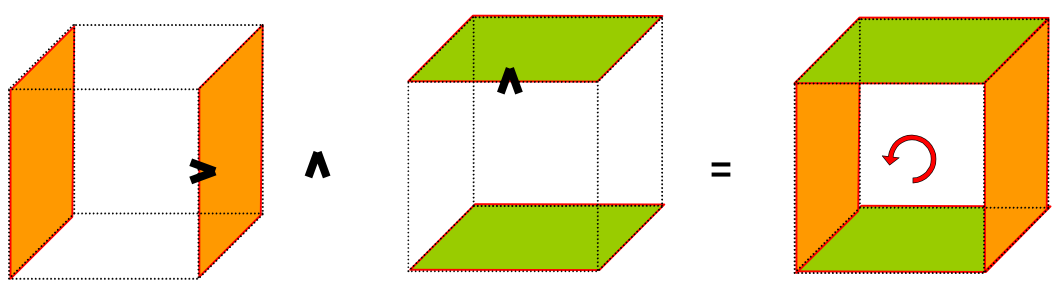
Representations

- ordered PAIR OF CLOSED CURVES
- directed cylinder ("a TUBE") with finite cross-sectional area

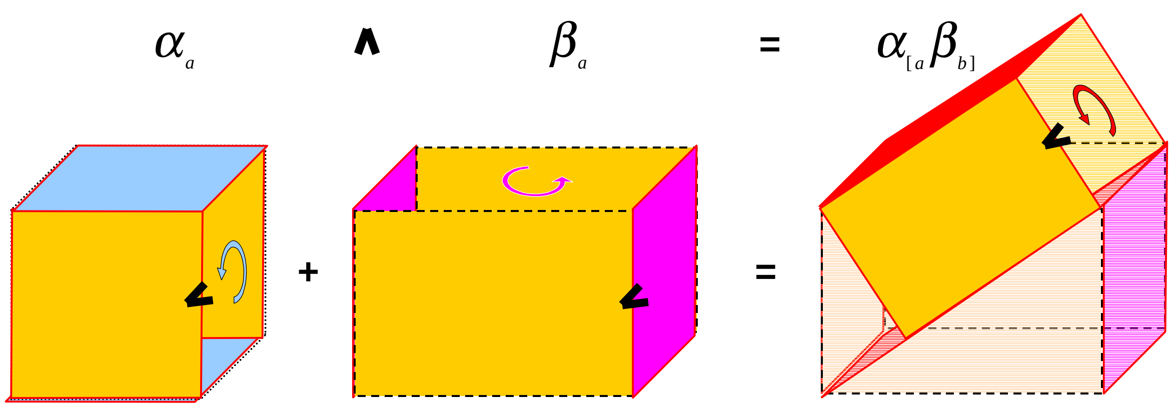
The cross-sectional area is *inversely*-proportional to its size.

Examples:

- magnetic induction B_{ab} [Weber/meter^2=Tesla]
  (magnetic flux per cross-sectional area) as in $\oiint_{\partial V} B_{ab} = 0$
- electric induction $\tilde{D}_{ab}$ [Coulomb/meter^2]
  (electric flux per cross-sectional area) as in $\oiint_{\partial V} \tilde{D}_{ab} = 4\pi q_{enclosed}$
- current density $\tilde{j}_{ab}$ [Ampere/meter^2]
  (charge flux per cross-sectional area) as in $\oint_{\partial A} \tilde{H}_{a} = \frac{\partial}{\partial t}\iint_A \tilde{D}_{bc} + 4\pi \iint_A \tilde{j}_{bc}$
- Poynting vector $\tilde{S}_{ab} = \frac{1}{4\pi} E_{[a}\tilde{H}_{b]}$ [Watt/meter^2]
  (energy flux per cross-sectional area)

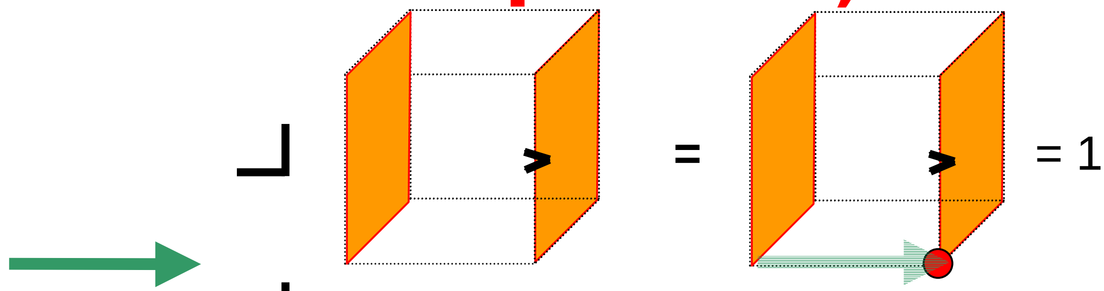


$\alpha_a \quad \wedge \quad \beta_a \quad = \quad \alpha_{[a}\beta_{b]}$

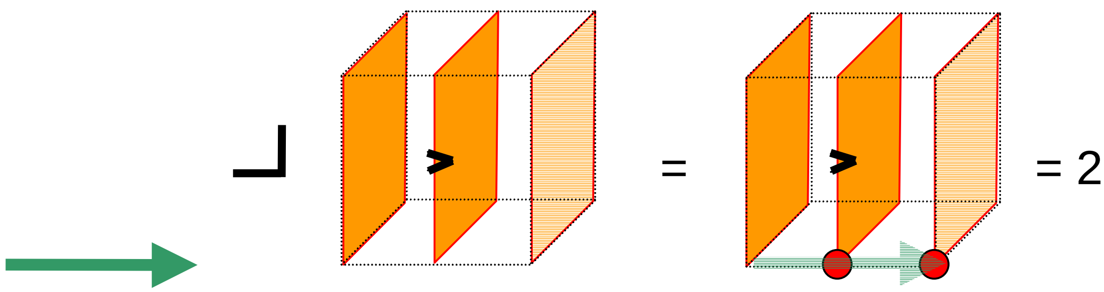


$\alpha_{[a}\beta_{b]} \quad + \quad \alpha_{[a}\gamma_{b]} \quad = \quad \alpha_{[a}(\beta_{b]} + \gamma_{b]})$

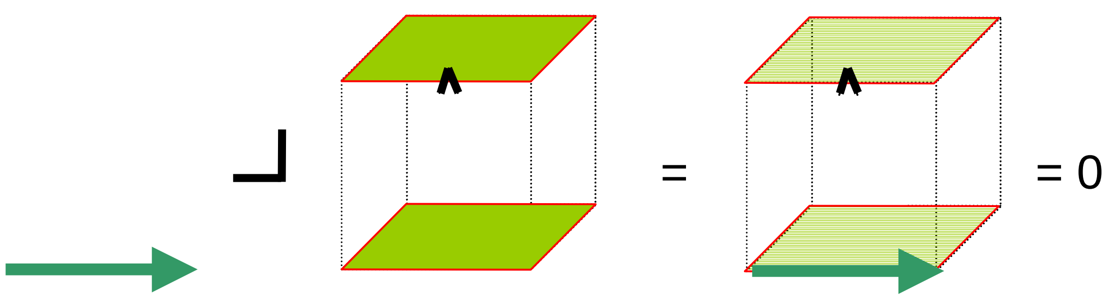
TRANSVECTION / INNER PRODUCT (nonmetrical "dot product") $\lrcorner$



$$V^a \lrcorner \; \omega_a = V^a \omega_a = 1$$



$$V^a \lrcorner \; 2\omega_a = V^a (2\omega_a) = 2$$



$$V^a \lrcorner \; \beta_a = V^a \beta_a = 0$$

In *Gravitation* (Misner, Thorne, Wheeler), this operation is described as counting the "bongs of a bell".

METRIC TENSOR g_{ab}

A metric tensor is a symmetric tensor that can be used to assign "magnitudes" to vectors.

$$\|V\|^2 = g_{ab}V^aV^b$$

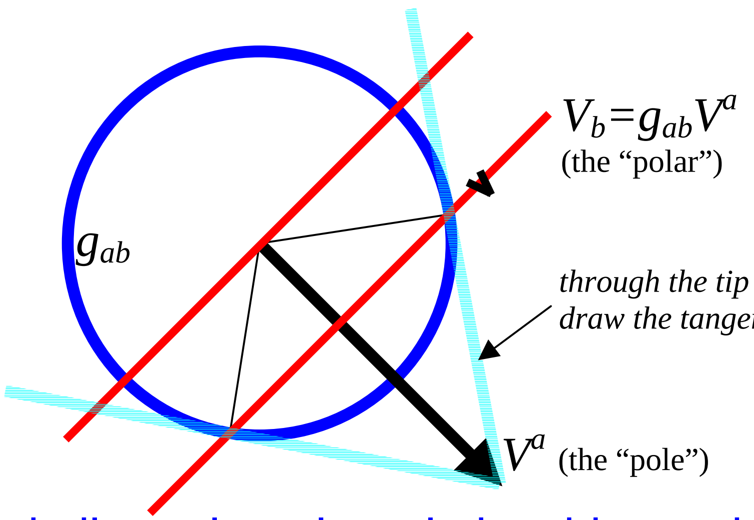
A metric tensor can also provide a rule to identify a vector with a unique covector. The vector and its covector are "duals" of each other with this metric.

Given a vector V^a, in the presence of a metric, we can form the combination $g_{ab}V^a$, which is a covector denoted by V_b. This is known as "index lowering", a particular move when performing "index gymnastics".

the Euclidean metric:

This construction is due to W. Burke, *Applied Differential Geometry*.

See also Burke, *Spacetime, Geometry, and Cosmology*.

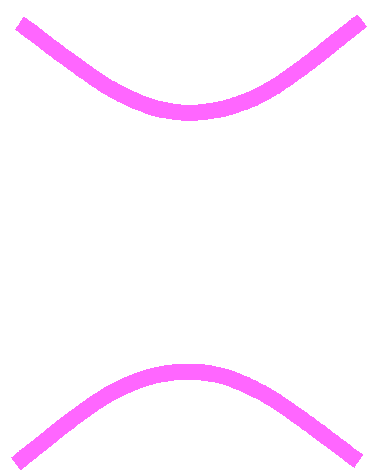


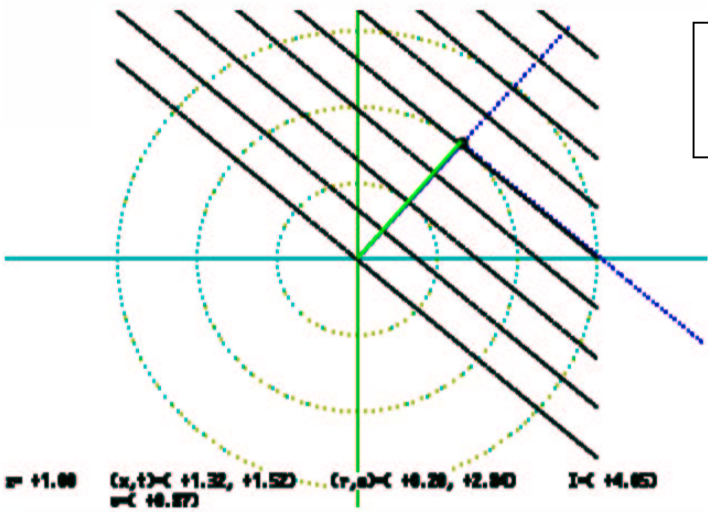
through the tip of the vectors, draw the tangents to the circle

A similar pole-polar relationship can be demonstrated for

Galilean

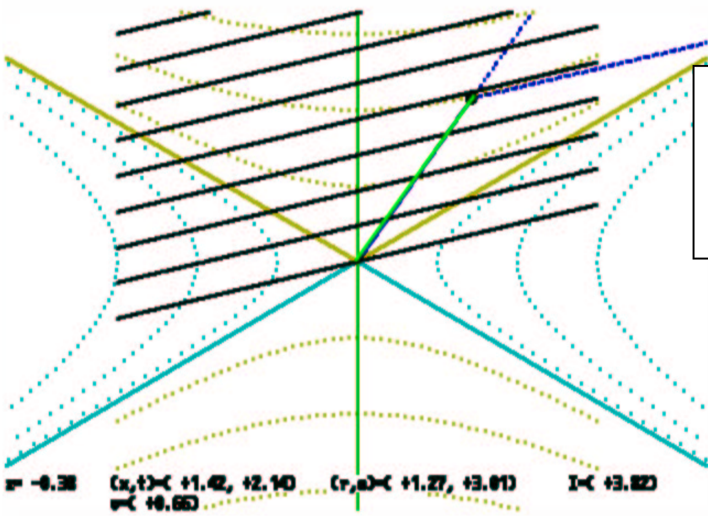
Minkowski



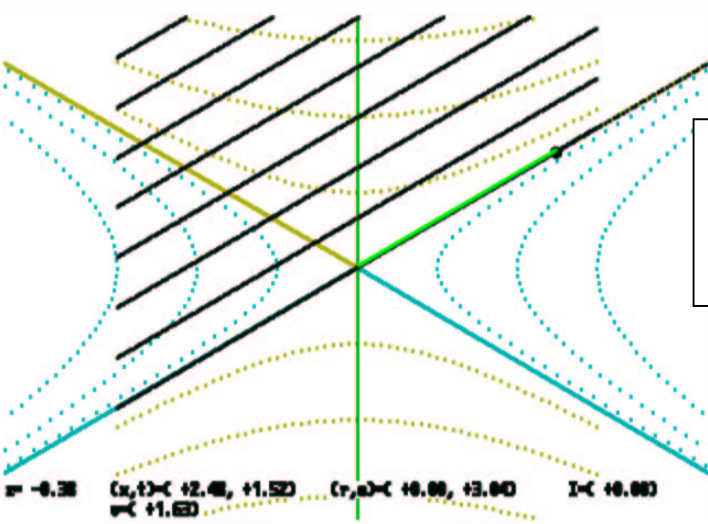


A vector of length 2
with a Euclidean metric.

Note that $V^a(g_{ab}V^b) = (\text{"length of } V^a\text{"})^2$.
Here $V^a(g_{ab}V^b) = 4$.



A timelike vector of
[about] length 2
with a Minkowski metric.



A lightlike vector has
zero length with a
Minkowski metric.

In three dimensional space, the following are not directed-quantities.

TRIVECTORS V^{abc}

Representations

- ordered TRIPLE OF VECTORS
- sensed regions ("a VOLUME") with finite size

The volume is proportional to its size.

Examples:

- volume V^{abc} [in meters^3] as in $V^{abc} = l^{[a} w^b h^{c]}$

THREE-FORMS γ_{abc}

Representations

- ordered TRIPLE OF COVECTORS
- cells ("a BOX") which contain a finite volume

The enclosed-volume is *inversely*-proportional to its size.

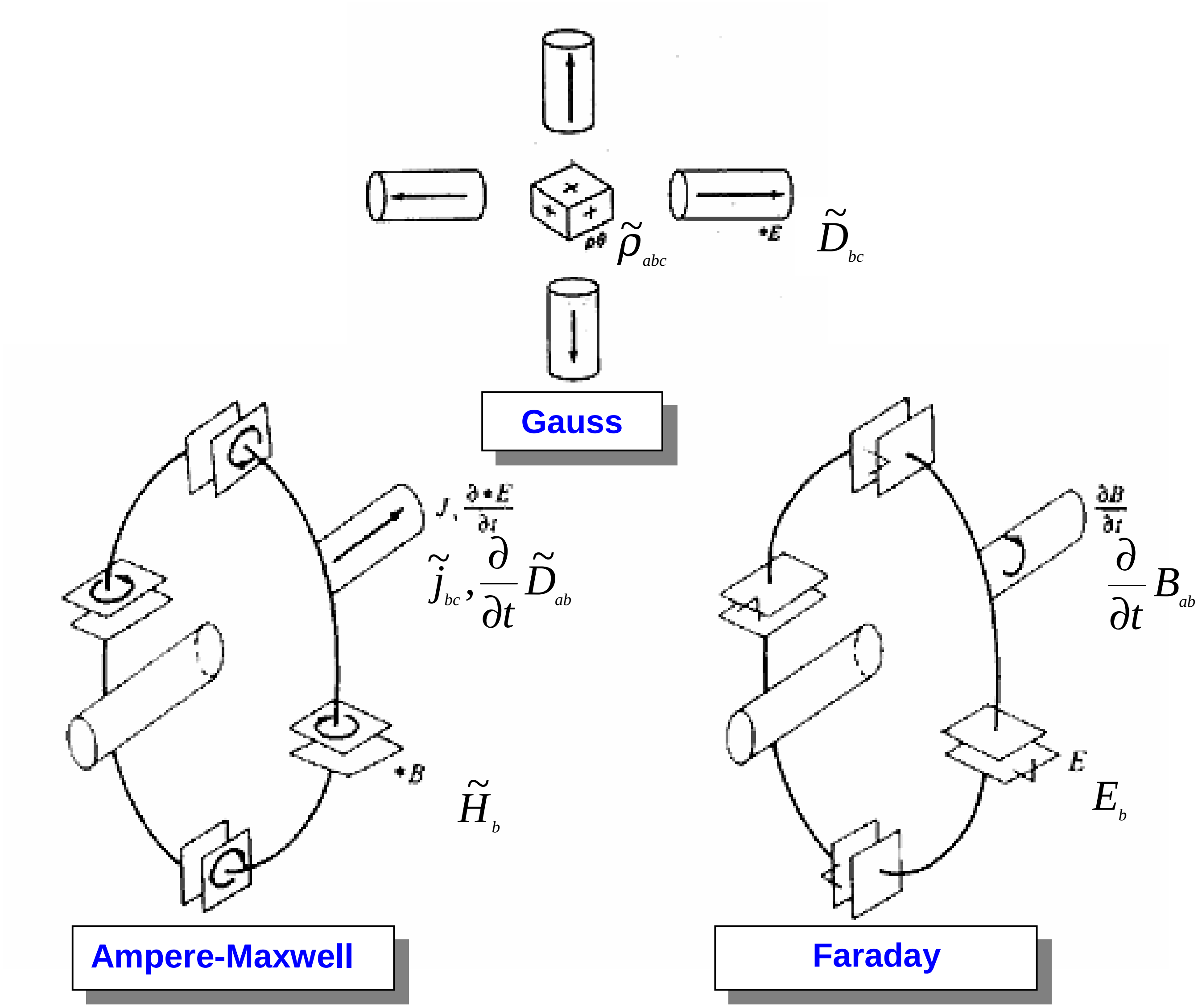
Examples:

- charge density $\widetilde{\rho}_{abc}$ [in Coulombs/meter^3] as in $q = \iiint_V \widetilde{\rho}_{abc}$
- energy density $\widetilde{u}_{abc}$ [in Joules/meter^3] as in $\widetilde{u}_{abc} = \frac{1}{8\pi} E_{[a} \widetilde{D}_{bc]}$

VOLUME FORM ε_{abc}

A volume form provides a rule to identify a vector with a unique two-form (in three dimensions), and vice versa. Vectors that are obtained from [ordinary] two-forms in this way are known as pseudovectors.

MAXWELL EQUATIONS



These diagrams are from
W. Burke, *Applied Differential Geometry*.

To see these rendered in three dimensions, visit the **VRML Gallery of Electromagnetism** at

physics.syr.edu/courses/vrml/electromagnetism

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