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Visualizing proper-time in Special Relativity

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Abstract

We present a new visualization of the proper-time elapsed along an observer’s worldline. By supplementing worldlines with light clocks, the measurement of space-time intervals is reduced to the “counting of ticks.” The resulting space-time diagrams are pedagogically attractive because they emphasize the relativistic view that “time is what is measured by an observer’s clock.”

I. Introduction

Einstein’s special relativity¹ forces us to revise our common-sense notions of time. Indeed, clocks in relative motion will generally disagree on the elapsed time interval measured between two meeting events. The discrepancy is practically undetectable for everyday relative speeds, but it is quite significant when the relative speeds are comparable to the speed of light. It is therefore necessary to distinguish these time intervals. Thus, we define for each observer his “proper-time” as the elapsed time interval measured by his clock. The goal of this paper is to find a physically intuitive visualization of this concept of proper-time.

Many textbooks^{2 3 4 5 6} introduce proper-time by analyzing the prop-

¹A. Einstein, “Zur Elektrodynamik bewegter Körper,” *Ann. Phys. (Leipzig)*, **17**, 891-921 (1905); reprinted as A. Einstein, “On the Electrodynamics of Moving Bodies” (1905) in *The Principle of Relativity*, by A. Einstein, H.A. Lorentz, H. Minkowski, and H. Weyl (Dover, New York, 1923), pp. 37-65.

²R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, vol. I (Addison-Wesley, Reading MA, 1963).

³D. Halliday, R. Resnick, J. Walker, *Fundamentals of Physics* (Wiley, New York, 1993); P.G. Hewitt, *Conceptual Physics* (Harper Collins, New York, 1993).

⁴E.F. Taylor and J.A. Wheeler, *Spacetime Physics* (W.H. Freeman, New York, 1966).

⁵N.D. Mermin, *Space and Time in Special Relativity* (McGraw-Hill, New York, 1968).

⁶A.P. French, *Special Relativity* (W.W. Norton, New York, 1968); T.A. Moore, *A Traveler’s Guide to Spacetime* (McGraw-Hill, New York, 1995).

agation of a light in a light clock,^{1 7 8 9 10 11} which consists of a pair of mirrors that face each other and are separated by a proper distance L . One “tick” of this clock is the duration of one round trip of a light ray bouncing back and forth between these mirrors. The analysis is usually done in the context of a simplified Michelson-Morley apparatus¹² whose arms may be regarded as light clocks. Unfortunately, most of these presentations^{2 3 5} work in moving frames of reference without making the connection to the space-time formulation, first introduced by Minkowski¹³ in 1907 and later extended by Einstein.¹⁴

Let us recall a quote from J.L. Synge:

...We have in the special theory of relativity the Minkowskian geometry of a flat 4-space with indefinite metric... Unfortunately, it has been customary to avoid this geometry, and to reason in terms of moving frames of reference, each with its own Euclidean geometry. As a result, intuition about Minkowskian space-time is weak and sometimes faulty....¹⁵

Indeed, when studying observers in relative motion, it is advantageous to draw a spacetime-diagram of the situation. However, we are immediately faced with an important question: “how does one know where to mark off the ticks of each clock?” More precisely, “given a standard of time marked on an observer’s worldline, how does one calibrate the same standard on the other observer’s worldline?”

One approach is to use the invariance of the speed of light to algebraically demonstrate the invariance of the spacetime-interval, from which the equation of a hyperbola arises.^{4 13} Then, for inertial observers that meet at a common event O , it can be shown that the corresponding ticks on their clocks [synchronized at event O] trace out hyperbolas on a space-time diagram. (See Figure 1.) Once this result is established, many of

⁷M. von Laue, *Jahrbuch der Radioaktivität und Elektronik* **14**, p. 263, (1917).

⁸R.F. Marzke and J.A. Wheeler, “Gravitation as Geometry—I: The Geometry of Space-Time and the Geometrodynamical Standard Meter,” in *Gravitation and Relativity*, edited by H.-Y. Chui and W.F. Hoffman (Benjamin, New York, 1964), pp. 40-64; C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman, New York, 1973);

⁹B. Bertotti, “The Theory of Measurement in General Relativity,” in *Evidence for gravitational theories: Proceedings of the International School of Physics “Enrico Fermi,”* edited by C. Møller (Academic Press, New York, 1962), pp. 174-201; S.A. Basri, “Operational Foundation of Einstein’s General Theory of Relativity,” *Rev. Mod. Phys.*, **37**, 288-315 (1965); J.L. Anderson and R. Gautreau, “Operational Approach to Space and Time Measurements in Flat Space,” *Am. J. Phys.*, **37**, 178-189 (1969).

¹⁰H. Arzeliés, *Relativistic Kinematics* (Pergamon Press, Oxford, 1966).

¹¹A.B. Arons, *Development of Concepts of Physics* (Addison-Wesley, Reading, Massachusetts, 1965).

¹²A.A. Michelson and E.W. Morley, “On the Relative Motion of the Earth and the Luminiferous Ether,” *Am. J. Sci.*, 3rd Ser., **34**, 333-345 (1887).

¹³H. Minkowski, “Space and Time” (1909) in *The Principle of Relativity* by H.A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, (Dover Publications, New York, 1923).

¹⁴A. Einstein, “On the Foundations of the Generalized Theory of Relativity and the Theory of Gravitation” (1914) in *The Collected Papers of Albert Einstein: Volume 4*, (Princeton University Press, Princeton, 1996).

¹⁵J.L. Synge, “Intuition, geometry, and physics in relativity,” *Annali di Matematica pura ed applicata*, **54**, 275-284 (1961).

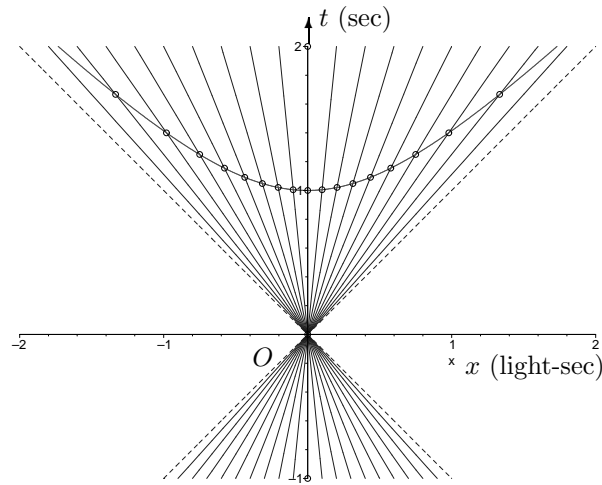


Figure 1: For inertial observers meeting at event O , the corresponding ticks of their clocks [synchronized at event O] trace out hyperbolas centered about event O .

the results of special relativity follow. This approach, however, is probably too sophisticated for a novice. Its connection with the more familiar (though provisional) physical concepts of time and space is not readily apparent.

In this paper, we connect the two approaches by drawing the spacetime diagram of the Michelson-Morley apparatus. Surprisingly, the only other spacetime diagram of the apparatus is a rough sketch in Synge’s *Relativity: The Special Theory*.¹⁶

The resulting diagram provides a visualization of proper-time which explicitly incorporates the principle of relativity and the invariance of the speed of light. The standard “effects” of time-dilation, length-contraction, and the relativity of simultaneity are easily inferred from the diagram. In addition, we show that standard calculations^{2 3 6 17 18 19} of the Clock Effect and the Doppler Effect can be reduced to the “counting of ticks.” We feel that the resulting diagrams are pedagogically attractive since they emphasize the relativistic view that “time is measured by an observer’s clock.”

In the last section, we will consider a simplified version of our clock, called the “longitudinal light clock.” Although this encodes fewer features than the full light clock, the longitudinal light clock is easy to draw manually.

In this paper, we have provided the detailed calculations used to draw

¹⁶J.L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1962), pp. 158-162.

¹⁷A. Schild, “The Clock Paradox in Relativity Theory,” *Am. Math. Monthly* **66**, 1-18 (1959).

¹⁸W. Rindler, *Essential Relativity* (Van Nostrand Reinhold, New York, 1977).

¹⁹G.F.R. Ellis and R.M. Williams, *Flat and Curved Space-Times* (Oxford University Press, Oxford, England, 1988).

the diagrams. However, we believe that one can first qualitatively construct the diagram for the novice, emphasizing the physical principles first. Then, for those interested, one can continue quantitatively with the analytical construction.

Following the standard conventions for spacetime diagrams, time runs upward on our spacetime diagrams. The scales of the axes are chosen so that light rays are drawn at 45 degrees.

II. A simplified Michelson-Morley apparatus

A. An apparatus at rest

Our simplified Michelson-Morley apparatus has a light source at the origin and two mirrors, each located a distance L along a *set of perpendicular arms*.

First, let us draw the spacetime diagram of the apparatus in its inertial rest-frame, called the “A-frame.” The coordinates (x, y, t) will be used to describe the events from this frame. Since relative motion will be taken to be along the x -axis, the mirror along the x -axis will be called the “longitudinal mirror” and the mirror along the y -axis will be called the “transverse mirror.”

The worldlines of A’s light source and mirrors are described parametrically by

$$\left. \begin{array}{l} \text{A's light source} \\ \text{A's transverse mirror} \\ \text{A's longitudinal mirror} \end{array} \right\} \begin{array}{l} \left\{ \begin{array}{l} x(t) = 0 \\ y(t) = 0 \end{array} \right\} \\ \left\{ \begin{array}{l} x(t) = 0 \\ y(t) = L \end{array} \right\} \\ \left\{ \begin{array}{l} x(t) = L \\ y(t) = 0. \end{array} \right\} \end{array} \quad (1)$$

Special relativity tells us that, in *all* inertial frames, light travels through the vacuum with speed c in all spatial directions, where the speed c has the value²⁰ of 2.99792458×10^8 m/s.

Let event O , with coordinates $(0, 0, 0)$, mark the emission of a flash of light from the source. One light ray emitted at event O reaches the transverse mirror at event

$$Y_A : \quad \left(0, \quad L, \quad \frac{L}{c} \right)$$

since light travels with speed c for a time L/c in order to reach the transverse mirror a distance L away. Its reflection is received back at the source at event

$$T_A : \quad \left(0, \quad 0, \quad 2\frac{L}{c} \right).$$

²⁰In 1983, the 17th *Conférence Générale des Poids et Mesures* defined the speed of light to be exactly 2.99792458×10^8 m/s and used this to define the meter.

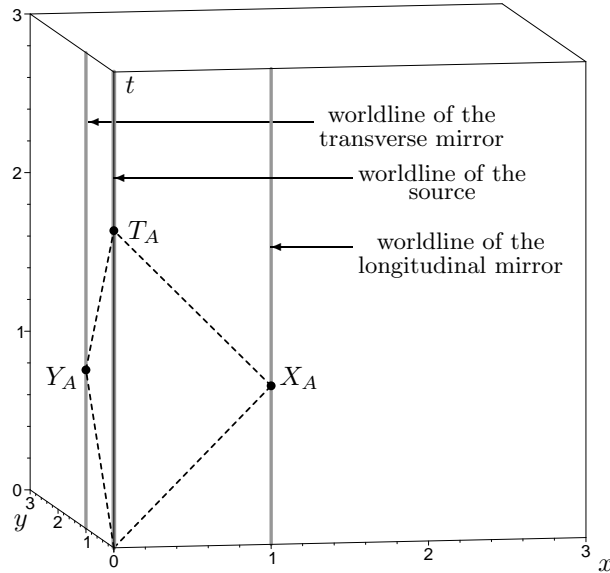


Figure 2: This is the spacetime diagram of a simplified Michelson-Morley apparatus in its rest-frame. (The x - and y -axes are marked in units of L . The t -axis is marked in units of L/c .)

Similarly, one light ray reaches the longitudinal mirror at event

$$X_A : \left(L, 0, \frac{L}{c} \right),$$

and its reflection is also received at event T_A . Hence, the two rays, emitted at event O and directed in different directions, are received at a common event T_A , whose coordinates are $(0, 0, 2L/c)$. (See Figure 2.)

If one arranges another light ray to be emitted upon reception (for example, by placing suitably oriented mirrors at the source), then this apparatus can serve as a simple clock—the light clock.

The reception event T_A marks one “tick” of this clock. This tick will be chosen to be the “standard tick.” The elapsed time logged by an observer sitting at the source is equal to the number of ticks multiplied by $2L/c$, the duration of one round trip of a light ray.²¹ If a finer scale of time is required, one can increase the resolution by choosing a smaller separation L . In addition, note that the reflection events X_A and Y_A , which are on mirrors equidistant from the source, can be regarded as “half-ticks” of this clock. We define these half-ticks to be “simultaneous events for this clock.”

²¹Implicitly, it is assumed that the impulses transferred to the mirrors upon reflection are negligible so that these mirrors maintain their separation from the source.

B. An apparatus in motion

Now, suppose an identical apparatus moves with spatial-velocity v parallel to the x -axis of the A-frame. This moving inertial-frame will be called the “B-frame,” and the coordinates (x', y', t') will be used to describe events from this frame. For simplicity, the origins of the primed and unprimed coordinate systems are taken to coincide at the emission event O . In addition, the corresponding spatial axes are assumed to be spatially-parallel²² within each inertial-frame.

Since B’s apparatus is identical to A’s, the worldlines of B’s light source and mirrors are described parametrically by

$$\left. \begin{array}{l} \text{B's light source} \\ \text{B's transverse mirror} \\ \text{B's longitudinal mirror} \end{array} \right\} \begin{array}{l} \left\{ \begin{array}{l} x'(t') = 0 \\ y'(t') = 0 \end{array} \right\} \\ \left\{ \begin{array}{l} x'(t') = 0 \\ y'(t') = L \end{array} \right\} \\ \left\{ \begin{array}{l} x'(t') = L \\ y'(t') = 0. \end{array} \right\} \end{array} \quad (2)$$

What does the spacetime diagram of this moving apparatus look like in the A-frame? In particular, what are the (x, y, t) coordinates of the worldlines of the moving apparatus and of the events X_B , Y_B , and T_B ?

Since B’s light source moves with velocity v in the x -direction, it is described as:

$$\text{B's light source} \quad \left\{ \begin{array}{l} x(t) = vt \\ y(t) = 0. \end{array} \right.$$

Similarly, since B’s transverse mirror also moves with velocity v in the x -direction, it is described as:²³

$$\text{B's transverse mirror} \quad \left\{ \begin{array}{l} x(t) = vt \\ y(t) = L. \end{array} \right.$$

Due to this mirror’s motion, the light ray from event O that meets this mirror must travel a longer distance in the A-frame. (See Figure 3.) Using the Pythagorean theorem, it is easily shown that this distance is $L/(1 - (v/c)^2)^{1/2}$. Thus, the reflection event on B’s transverse mirror is

$$Y_B : \left(v \frac{L}{c} \frac{1}{\sqrt{1 - (v/c)^2}}, \quad L, \quad \frac{L}{c} \frac{1}{\sqrt{1 - (v/c)^2}} \right).$$

²²This is the usual convention for “aligning” the spatial axes. In special relativity, the x - and x' - axes are *not* parallel in *spacetime*. However, in the A-frame, the spatial-projection of the x' -axis *is* parallel to the x -axis, and conversely for the B-frame. By contrast, in Galilean relativity, the corresponding spatial axes are parallel in both the spatial and spacetime senses.

²³Strictly speaking, we should prove that, for a moving object, lengths along its transverse directions are unchanged. We refer the reader to Feynman (footnote 2), Mermin (footnote 5), and Arons (footnote 11) for a symmetry argument.

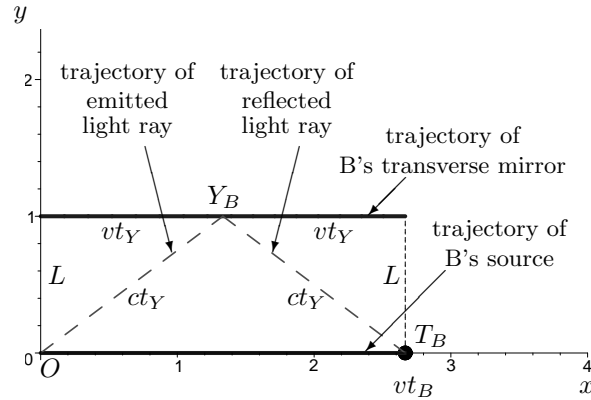


Figure 3: On A's xy -plane, the spatial trajectories of B's transverse arm and its associated light rays are drawn. The marked dot corresponds to the spatial coordinates of B's first "tick," which occurs after an elapsed time t_B in the A-frame. Let t_Y be the elapsed time for a light ray from event O to reach B's transverse mirror. Using the Pythagorean theorem, it can be shown that $t_Y = (L/c)(1 - (v/c)^2)^{-1/2}$. By symmetry, it follows that $t_B = (2L/c)(1 - (v/c)^2)^{-1/2}$, which is longer than the duration of A's tick, $(2L/c)$. This is called the "time dilation" effect. (The x - and y -axes are marked in units of L .)

Using a similar argument, the reflected ray is received by B's source at:

$$T_B : \left(v \frac{2L}{c} \frac{1}{\sqrt{1 - (v/c)^2}}, \quad 0, \quad \frac{2L}{c} \frac{1}{\sqrt{1 - (v/c)^2}} \right).$$

Note that, in the A-frame, event T_B ("the first tick of the moving clock") occurs later than T_A ("the first tick of the stationary clock"). This is the "time dilation" effect.

Now, let us consider the longitudinal mirror. Recall that the reflected light rays for A's apparatus were received by A's source at a common event, namely, T_A . According to the principle of relativity, the apparatuses cannot distinguish their states of inertial motion. Thus, the reflected light rays for B's apparatus must also be received by B's source at a common event, here, T_B .

The required reflection event X_B on B's longitudinal mirror is determined as follows. Event X_B is the intersection on the xt -plane of the forward-directed light ray from event O and the backward-directed light

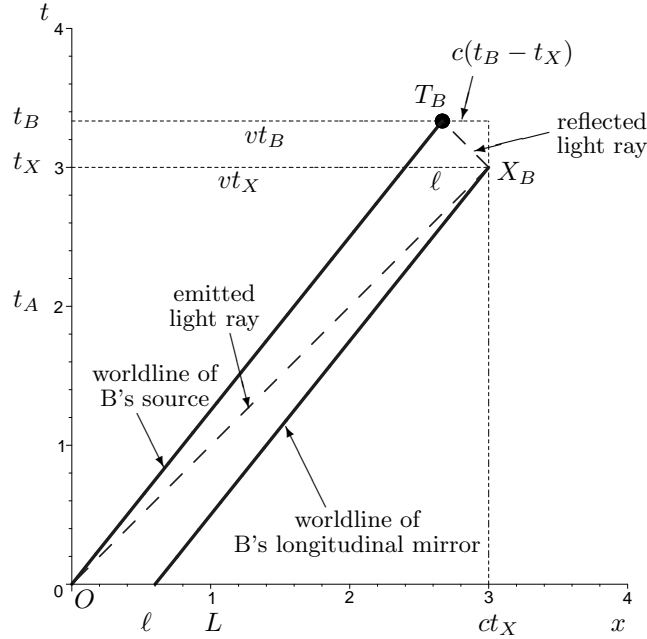


Figure 4: On A’s xt -plane, the worldlines of B’s longitudinal arm and its associated light rays are drawn. The marked dot corresponds to B’s first “tick.” Let ℓ be the apparent length of B’s longitudinal arm. Let t_X be the elapsed time for a light ray from event O to reach B’s longitudinal mirror. Since $ct_X = vt_B + c(t_B - t_X)$ and $t_B = (2L/c)(1 - (v/c)^2)^{-1/2}$, it can be shown that $t_X = (L/c)((1 + v/c)/(1 - v/c))^{1/2}$. Furthermore, since $ct_X = vt_X + \ell$, it follows that $\ell = L(1 - (v/c)^2)^{1/2}$, which is shorter than the proper length L of A’s identical apparatus. This is the “length contraction” effect. (The x -axis is marked in units of L . The t -axis is marked in units of L/c .)

ray toward event T_B . (See Figure 4.) After a little algebra, the reflection event on B’s longitudinal mirror is determined to be

$$X_B : \left(c \frac{L\sqrt{1 - (v/c)^2}}{c - v}, \quad 0, \quad \frac{L\sqrt{1 - (v/c)^2}}{c - v} \right).$$

Thus, B’s longitudinal mirror is described by:

$$\text{B's longitudinal mirror} \quad \begin{cases} x(t) = vt + L\sqrt{1 - (v/c)^2} \\ y(t) = 0. \end{cases}$$

Note that, in the A-frame, the length of B’s longitudinal arm is $L(1 - (v/c)^2)^{1/2}$, which is shorter than its proper length L . This is the “length contraction” effect.

This completes the construction of B’s apparatus.

As a check, note that these results can be obtained directly from the Lorentz transformation:

$$\left. \begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - (v/c)^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - (v/c)^2}} \\ y' &= y \end{aligned} \right\} \quad (3)$$

For instance, given the worldlines for B's apparatus in (x', y', t') -coordinates (Equation 2), expressions for x and y as functions of t can be obtained:

$$\left. \begin{aligned} \text{B's source} & \quad \left\{ \begin{aligned} x(t) &= vt \\ y(t) &= 0 \end{aligned} \right. \\ \text{B's transverse mirror} & \quad \left\{ \begin{aligned} x(t) &= vt \\ y(t) &= L \end{aligned} \right. \\ \text{B's longitudinal mirror} & \quad \left\{ \begin{aligned} x(t) &= vt + L\sqrt{1 - (v/c)^2} \\ y(t) &= 0. \end{aligned} \right. \end{aligned} \right\} \quad (4)$$

Similarly, given the (x', y', t') -coordinates of B's tick and half-ticks, the (x, y, t) -coordinates of X_B , Y_B , and T_B can be obtained.

These results are summarized in Figure 5.

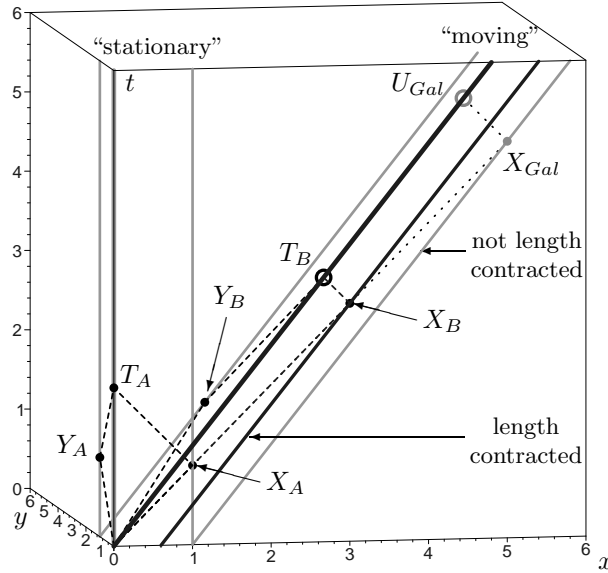


Figure 5: This is A’s spacetime diagram of an identical apparatus moving with velocity $v = 0.8c$ along A’s x -axis. Observe that without length contraction, the light rays reflected by the moving mirrors are not received simultaneously (at T_B) by the moving source.

Clearly, the length-contraction factor $(1 - (v/c)^2)^{1/2}$, which is missing in the Galilean transformations,²⁴ enforces the requirement that the reflection occur at event X_B so that the reflected light ray is received at event T_B . Without length contraction, the reflection would have occurred at event X_{Gal} , and the reception event U_{Gal} would have occurred at the source *after* event T_B . Such a result would violate the principle of relativity since one’s inertial state of motion could now be detected. (See Figure 5.) Indeed, the apparatus of Michelson and Morley was used to measure the predicted time difference between events U_{Gal} and T_B , in accordance with the Galilean transformations. However, no time difference was experimentally observed.^{12 25}

In addition to the time-dilation and length-contraction effects, note that the events X_B and Y_B , which are defined to be simultaneous according to B’s clock, are not simultaneous according to A’s clock. This is the “relativity of simultaneity.”

For clarity, it is useful to introduce the standard abbreviations. Let γ

²⁴The Galilean transformations are $t' = t$, $x' = x - vt$, $y' = y$.

²⁵W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, Massachusetts, 1955).

denote the time-dilation factor

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}, \quad (5)$$

and let k denote the Doppler-Bondi factor^{19 26}

$$k = \sqrt{\frac{1 + (v/c)}{1 - (v/c)}}. \quad (6)$$

With these abbreviations, the coordinates of X_B , Y_B , and T_B can be expressed as

$$\begin{aligned} X_B : & \quad \left(kL, \quad 0, \quad k\frac{L}{c} \right) \\ Y_B : & \quad \left(\gamma v\frac{L}{c}, \quad L, \quad \gamma\frac{L}{c} \right) \\ T_B : & \quad \left(2\gamma v\frac{L}{c}, \quad 0, \quad 2\gamma\frac{L}{c} \right). \end{aligned}$$

For the examples used throughout this paper, the B-frame moves with velocity $v = 0.8c$ relative to the A-frame. For this choice, we have $\gamma = 5/3$ and $k = 3$.

III. Circular Light Clocks

A. A generalized apparatus

Generalizing the analysis of the last section, it is easy to see that:

With *any* relative orientation of the arms one would obtain the same results:

1. Light rays emitted by the source at event O to mirrors a distance L away would be received back at the source at a time $2L/c$ later.
2. The reflection events at the mirrors are simultaneous according to that source.

So, instead of a *pair* of equidistant mirrors, consider a whole collection of mirrors placed inside a circle [generally, a sphere] of radius L . Henceforth, this will be called the “circular light clock.”²⁷ What would the spacetime diagram of this light clock look like?

In this case, one would have a hollow worldtube to describe the collection of mirrors for each clock. In addition, for each tick of a given clock, one would draw the portion of its light cone contained inside the clock’s

²⁶H. Bondi, *Relativity and Common Sense* (Dover, New York, 1962); H. Bondi, “Some Special Solutions of the Einstein Equations,” in *Lectures on General Relativity: Brandeis 1964 Summer Institute on Theoretical Physics, vol. 1*, by A. Trautman, F.A.E. Pirani, and H. Bondi (Prentice-Hall, Englewood Cliffs, New Jersey, 1965), pp. 375-459.

²⁷After writing this paper, I learned that this is sometimes called the “Fokker clock.” A.D. Fokker, “Accelerated spherical light wave clocks in chronogeometry,” *Nederl. Akad. Wetensch. Proc. Ser. B* **59**, 451-454 (1956).

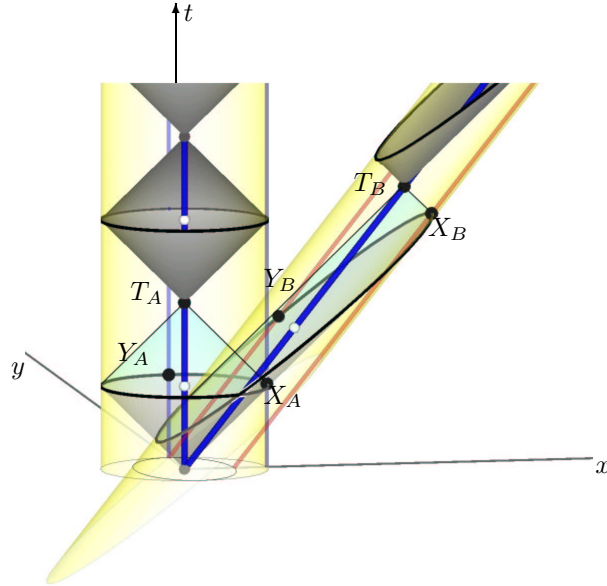


Figure 6: Two circular light clocks in relative motion. For each light clock, the intersection of the worldtube and the light cones from two consecutive ticks is a circle of simultaneous events for that clock. The white dots represent events at the source that are simultaneous with the corresponding circle of intersection. Note that the “moving” circular mirror is length-contracted in this inertial frame.

worldtube. These cones represent events in spacetime traced out by the collection of light rays that reflect off the mirrors from tick to tick. (See Figure 6.)

In particular, the stationary light clock will be drawn with a circular cross-section. The moving one will be drawn tilted with an elliptical cross-section since it is length-contracted in the direction of relative motion. Then, given a starting emission event on the axis of each worldtube, one traces out the paths of the light rays and their reflections, which are drawn upward with a slope of 45 degrees in this diagram. For simplicity, the starting emission event O is taken to be the intersection of the two sources. (Refer again to Figure 6.)

We now make a series of observations.

Note that the original pair of reflection events X_A and Y_A (and X_B and Y_B , respectively) are among the events on the circle of mutual intersections of the worldtube of the light clock and the light cones of its zeroth and first ticks. We extend our definition of simultaneity according to this clock to that circle of events. In fact, one can extend this simultaneity to events on the unique [hyper]plane that contains this circle [respectively, sphere]. Physically, this [hyper]plane represents “all of space at a particular instant

for this light clock.” Again note that the events that are simultaneous to this light clock are generally different from those events determined to be simultaneous by the other light clock.

With this notion of simultaneity, the light cones can be interpreted in a complementary way. For each light clock, a simultaneous “slice” of its cones represents a circular [respectively, spherical] wavefront traveling at the speed of light. Hence, the light cone can also be interpreted as a sequence of wavefronts traveling at the speed of light.²⁸

B. Visualizing proper-time

By continuing this light-clock construction along each inertial observer’s worldtube, an accurate visual representation of the proper time elapsed for each observer is obtained. (See Figure 7.) From the diagram, however, it may not be evident that the two observers are equivalent.

First, we demonstrate their symmetry with the Doppler Effect. Suppose these inertial observers emit light signals at one-tick intervals. From the diagram, each observer receives those signals from the other observer at three-tick intervals. In other words, the received frequency is one-third of the original frequency, in accordance with the Doppler effect for two observers separating with speed $v = 0.8c$. In general, a light ray emitted by a source at its first tick after separation reaches the receiver at the receiver’s k th tick after separation, where k is the Doppler-Bondi factor defined in equation 6. This is the basis of the Bondi k -calculus.^{19 26 29 30} (See Figure 8.)

Next, we demonstrate their symmetry with the time-dilation effect. Consider the signal emitted at the first tick. As just noted, this signal is received by the receiver at his third tick. According to the source of that signal, that distant reception event is simultaneous with his fifth tick. In other words, the apparent elapsed time assigned to a distant event is five-thirds as long as the proper elapsed time measured by the inertial observer who visits that distant event. In general, the apparent elapsed time assigned to a distant event is γ times as long as the the proper elapsed time measured by the inertial observer who meets that distant event, where γ is the time-dilation factor defined in equation 5. (Refer again to Figure 8.)

In addition, the source measures the apparent distance to that reception event to be $vt_{\text{apparent}} = v \cdot (\gamma k \text{ ticks})$. For $v = 0.8c$, this is

$$v\gamma k \text{ ticks} = (0.8c) \left(\frac{5}{3}\right) (3)(1 \text{ tick}) = 4 \text{ ticks} \cdot c = 4 \text{ light-ticks.}$$

Since L is the radius of the worldtube and $1 \text{ tick} = (2L/c)$, this distance can be expressed as “four worldtube *diameters*.” This suggests the

²⁸H.M. Schey, “Expanding Wavefronts in Special Relativity: A Computer-Generated Film,” Am. J. Phy. **37**, 514-519 (1969); W. Moreau, “Wave front relativity,” Am. J. Phy. **62**, 426-429 (1994).

²⁹N.D. Mermin, “An introduction to space-time diagrams,” Am. J. Phy. **65**, 477-486 (1997).

³⁰L. Marder, *Time and the Space-Traveller* (University of Pennsylvania Press, Philadelphia, 1971).

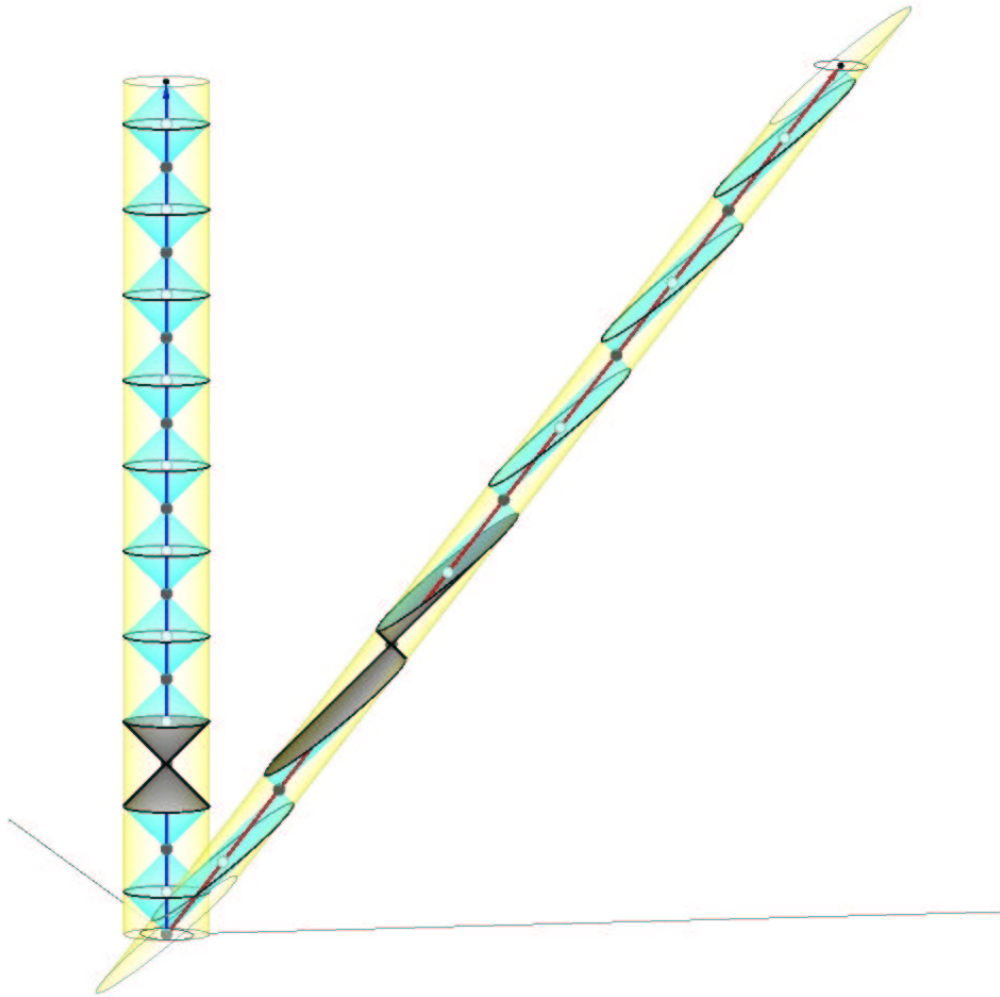


Figure 7: Two inertial observers carry circular light clocks. For each clock, the portion of the second tick's light cone inside the worldtube is highlighted.

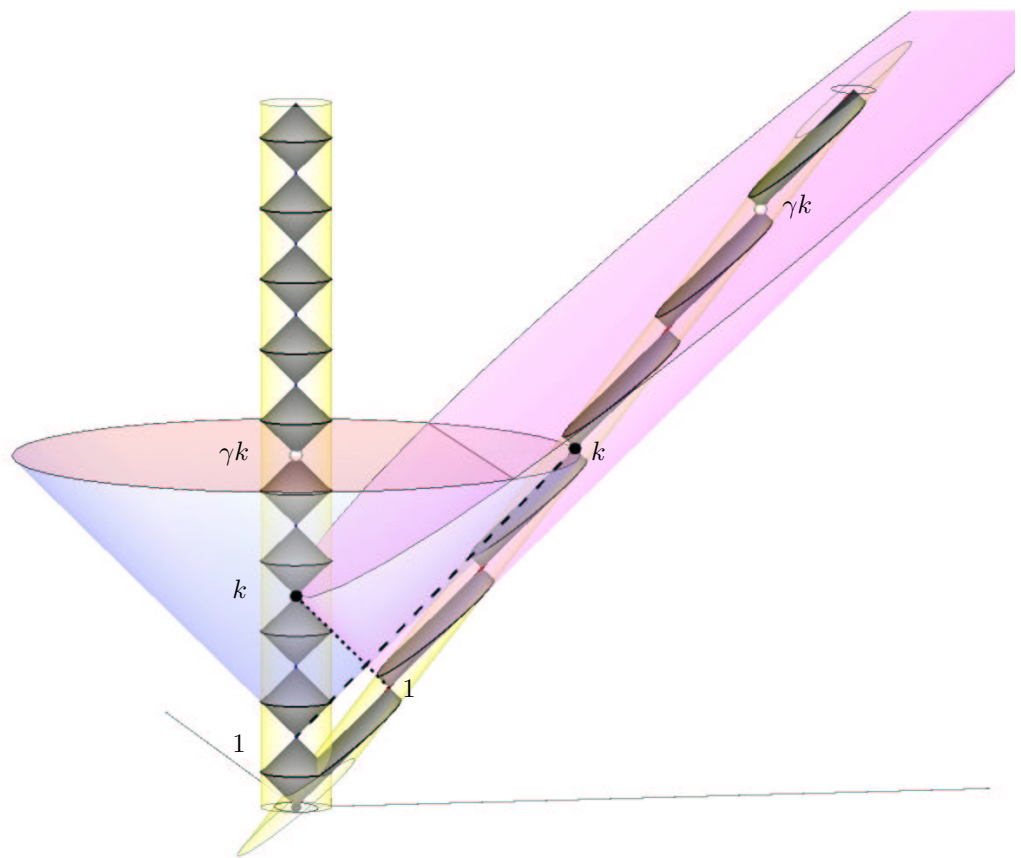


Figure 8: Symmetry of the observers: The Doppler and Time-Dilation effects. For a relative speed of $v = 0.8c$, we have $\gamma = 5/3$ and $k = 3$.

diagrams in Figure 9. Of course, this is just the calculation of the square-interval in terms of the temporal and spatial coordinates

$$\begin{aligned}
 (\text{proper-time})^2 &= (\text{apparent-time})^2 - (\text{apparent-distance}/c)^2 \quad (7) \\
 (3 \text{ ticks})^2 &= (5 \text{ ticks})^2 - (4 \text{ ticks})^2,
 \end{aligned}$$

which can be regarded as the spacetime version of the Pythagorean theorem. Note that for a constant value of the proper-time, the admissible pairs of temporal and spatial coordinates locate events on a hyperbola.

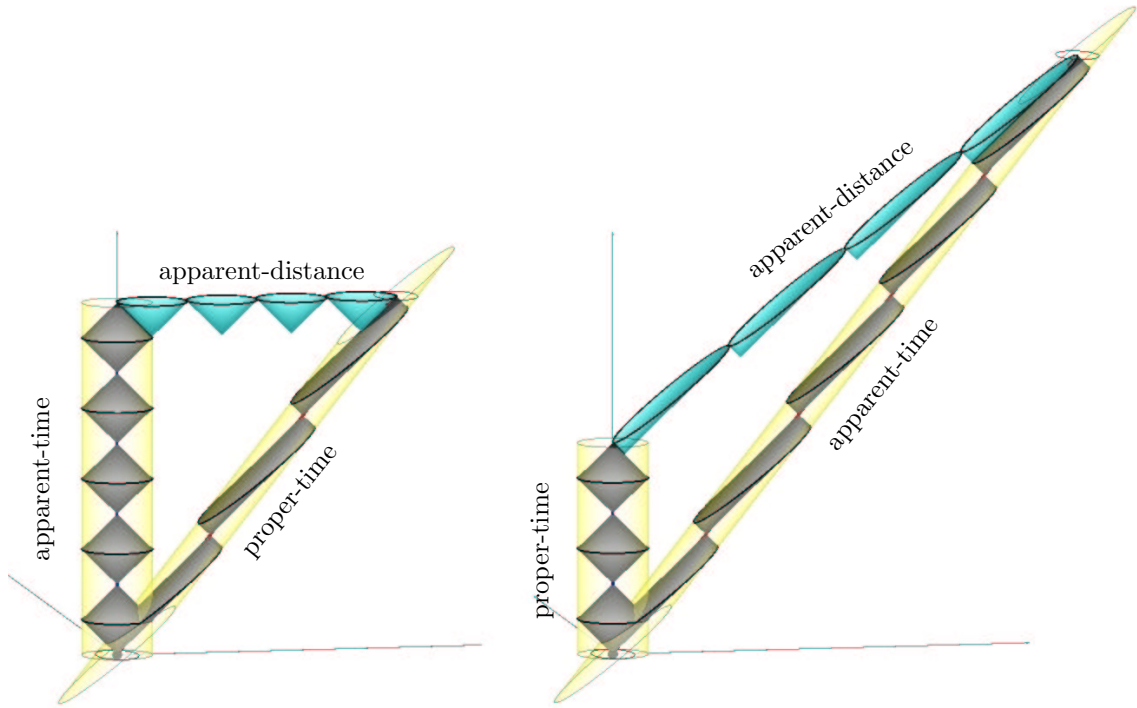


Figure 9: Symmetry of the observers: The spacetime analogue of the Pythagorean Theorem: $(\text{proper-time})^2 = (\text{apparent-time})^2 - (\text{apparent-distance}/c)^2$.

C. The Clock Effect

With this pictorial device, we present a visual representation of the Clock Effect. (See Figure 10.)

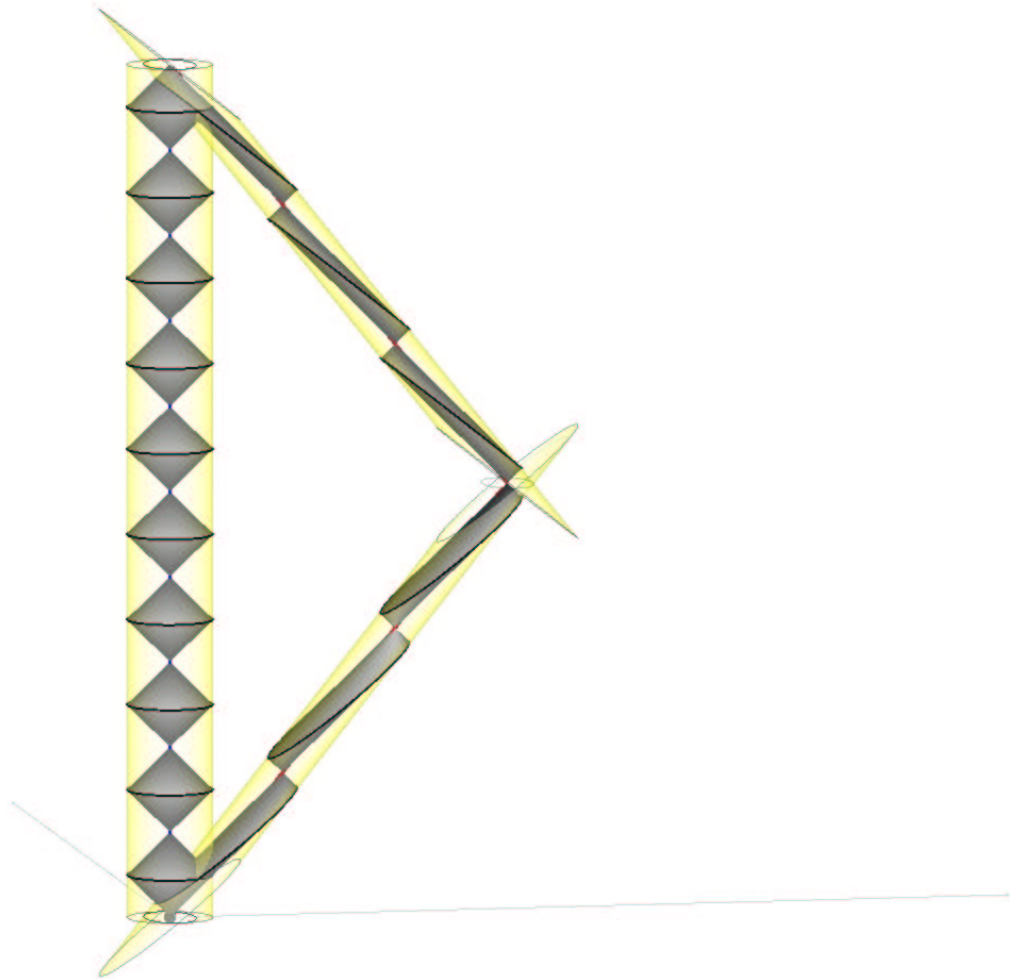


Figure 10: The Clock Effect. A non-inertial observer travels away with velocity $v = 0.8c$ for 3 ticks, the returns with velocity $v = -0.8c$ for another 3 ticks. Between the departure and reunion events, he has logged 6 ticks for his entire trip, whereas the inertial observer has logged 10 ticks.

From the diagram, the non-inertial observer travels away with velocity $v = 0.8c$ for 3 ticks, the returns with velocity $v = -0.8c$ for another 3

ticks. Between the departure and reunion events, he has logged 6 ticks for his entire trip. On the other hand, the inertial clock logs

$$\frac{1}{\sqrt{1 - (0.8c/c)^2}}(\text{first 3 ticks}) + \frac{1}{\sqrt{1 - (-0.8c/c)^2}}(\text{second 3 ticks}) = 10 \text{ ticks}$$

between the same departure and reunion events.

Clearly, the diagram reveals that more time elapses for the inertial observer that meets both events.

In addition, the two observers are certainly inequivalent. The kink in the non-inertial worldline causes the sequence of simultaneous events to change discontinuously, leading to the apparent break in the non-inertial worldtube. This is not to say that the non-inertial worldtube actually breaks. Rather, it is an artifact of how the diagram was drawn. In order to draw the true worldtube, a more careful analysis with a detailed model of the apparatus is needed. We refer the reader to some articles on the Clock Effect that discuss this kink in the non-inertial observer’s worldline.^{10 19}
30

D. A brief summary

Let us summarize the logical development up to this point.

Given the simplified Michelson-Morley apparatus in relative motion, the invariance of the speed of light (so that *all* light rays are drawn at an angle of 45 degrees) is used to draw the light rays associated with the perpendicular arm. Invoking the principle of relativity (so that the duration of the round-trip defines the same standard tick), we deduce the effect of time dilation. Again using the invariance of the speed of light, we draw the light rays associated with the parallel arm. Again invoking the principle of relativity, we deduce the effect of length contraction.

Generalizing these results to arbitrary directions, we obtain the circular light clock. By continuing this construction along a piecewise-inertial worldline, we obtain a visual representation of the proper-time elapsed along that worldline.

IV. Longitudinal Light Clocks

Let us now consider a simplified two-dimensional version of the light clock diagram. Consider the worldlines of the source and of one longitudinal mirror, that is, the longitudinal light clock. As before, the moving light clock appears length-contracted in the direction of relative motion. In this case, we have the following diagrams. (See Figures 11 and 12.)

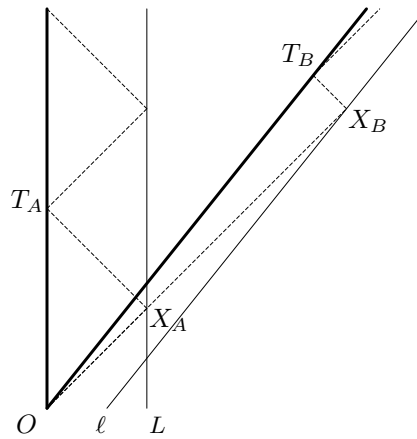


Figure 11: Two longitudinal light clocks. The apparent length ℓ of the moving apparatus was shown to be $L(1 - (v/c)^2)^{1/2}$.

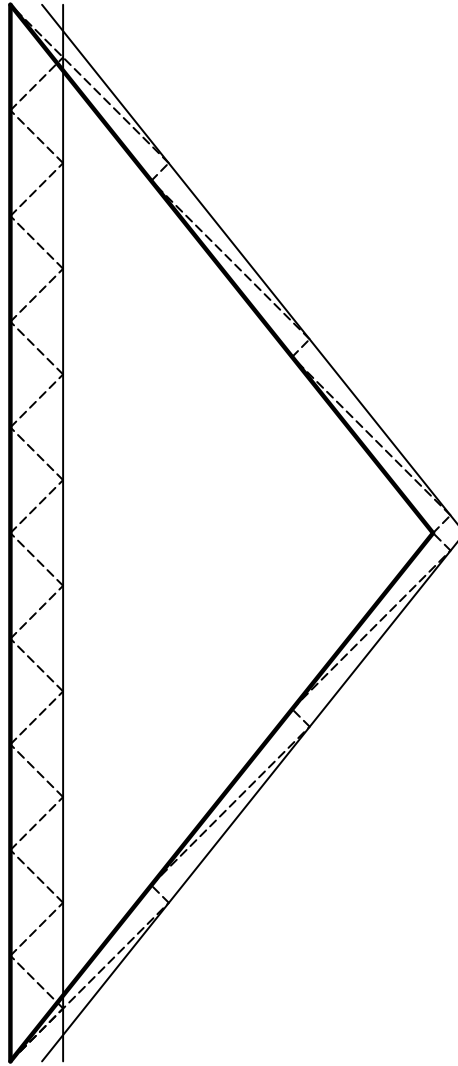


Figure 12: The Clock Effect with the longitudinal light clocks.

Although these figures are much easier to draw, it is unfortunate that the role of length contraction appears here so prominently. Recall that event T_B was determined using the invariance of the speed of light and the principle of relativity, which required that the transverse and longitudinal reflections be received simultaneously at the source. However, without the transverse direction, the role of invariance may not be evident.

In this section, we will draw attention to a certain geometric property of this diagram and use it to emphasize instead the “invariance of the spacetime interval”.

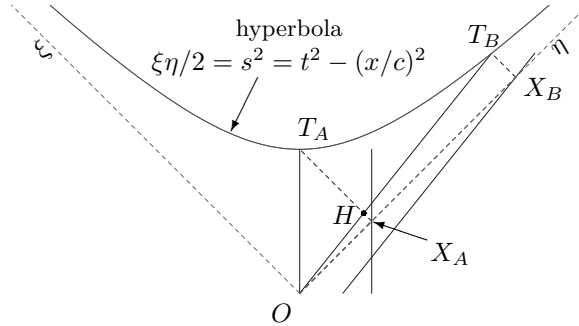


Figure 13: Since T_A and T_B are points of this hyperbola, the triangles $\triangle OX_A T_A$ and $\triangle OX_B T_B$ have the same area. With $(OX_B)/(OX_A) = k$, where k is Doppler-Bondi factor, a useful corollary is that $(X_A H)/(X_A T_A) = 1/k^2$. With these facts, given O , T_A , X_A , and H , one can easily determine X_B and T_B .

A. An invariant area

Refer to the diagram of two longitudinal light clocks. (See Figure 13.)

Consider the triangles $\triangle OX_A T_A$ and $\triangle OX_B T_B$, formed from the timelike intervals from event O to the first ticks and their associated light rays. Since, on a spacetime diagram, events T_A and T_B are at equal intervals from event O , they lie on a rectangular hyperbola asymptotic to the light cone of event O . From this, it can be shown that these triangles (which are related by Lorentz transformations) have the same area. The following calculation reveals that this area is proportional to the square-interval of one tick.³¹

Consider the segment drawn from the emission event O to any event P with coordinates (x, t) . Regard that segment as the hypotenuse of a Euclidean right triangle whose sides are parallel to the light cone of event O . The legs of this triangle have measure

$$\xi = \frac{t - x/c}{\sqrt{2}} \quad \eta = \frac{t + x/c}{\sqrt{2}}.$$

These are called the Dirac light-cone coordinates^{31 32} in the A-frame. In these coordinates, the Euclidean area of this triangle is simply $\xi\eta/2$, which

³¹S. Daubin, “A Geometrical Introduction to Special Relativity,” *Am. J. Phys.* **30**, 818-824 (1962); D. Bohm, *The Special Theory of Relativity* (W.A. Benjamin, New York, 1965), pp. 152-153; Y.S. Kim and M.E. Noz, “Dirac’s light-cone coordinate system,” *Am. J. Phys.* **50** (8), 721-724 (1982); N.D. Mermin, “Space-time intervals as light rectangles,” *Am. J. Phys.* **66**, 1077-1080 (1998).

³²P.A.M. Dirac, “Forms of Relativistic Dynamics,” *Rev. Mod. Phys.* **21**, 392 (1949); L. Parker and G.M. Schmieg, “Special relativity and diagonal transformations,” *Am. J. Phys.*

is equal to $(t^2 - (x/c)^2)/4$. That is, the Euclidean area of that triangle is equal to one-fourth of the square-interval.

Let us explicitly verify that this area is invariant under Lorentz transformations. Using Equation 3, the light-cone coordinates in the B-frame are

$$\begin{aligned}\xi' &= \frac{t' - x'/c}{\sqrt{2}} = \gamma \left(1 + \frac{v}{c}\right) \left(\frac{t - x/c}{\sqrt{2}}\right) = k \left(\frac{t - x/c}{\sqrt{2}}\right) = k\xi \\ \eta' &= \frac{t' + x'/c}{\sqrt{2}} = \gamma \left(1 - \frac{v}{c}\right) \left(\frac{t + x/c}{\sqrt{2}}\right) = k^{-1} \left(\frac{t + x/c}{\sqrt{2}}\right) = k^{-1}\eta,\end{aligned}$$

where k is Doppler-Bondi factor defined in Equation 6. Thus, the quantity $\xi\eta/2$ is Lorentz invariant.

B. A simple construction

This invariance of the area can be used to draw the light clock of a moving observer, given the light clock of the stationary observer and the worldline of the moving observer.

For event T_A , the legs of its associated triangle in the A-frame are $\eta_A = (OX_A)$ and $\xi_A = (X_AT_A)$. Similarly, for event T_B , the legs of its associated triangle in the A-frame are $\eta_B = (OX_B)$ and $\xi_B = (X_AT_B)$. So, we have

$$\frac{(OX_B)}{(OX_A)} = \frac{\eta_B}{\eta_A} = \frac{k\eta'_B}{\eta_A} = \frac{k \left(\frac{2L}{c\sqrt{2}}\right)}{\left(\frac{2L}{c\sqrt{2}}\right)} = k.$$

Now, refer again to figure 13. Using the similarity of triangles $\triangle OX_AH$ and $\triangle OX_BT_B$ and the fact that $(OX_B)/(OX_A) = k$, we obtain the useful corollary that

$$\frac{(X_AH)}{(X_AT_A)} = \frac{(X_BT_B)/k}{k(X_BT_B)} = 1/k^2.$$

With this property, we can now draw the longitudinal light clock with the emphasis on the invariance of the interval, rather than on length contraction.

Given the standard tick for the stationary observer (events O , T_A , and X_A) and the worldline of a moving observer (line OH), one can determine the standard tick for the moving observer (events O , X_B and T_B) as follows. Measure $(X_AH)/(X_AT_A)$. [For a classroom activity, one might use a sheet of graph paper with its axes aligned with the forward light cone of event O .] Using the corollary $(X_AH)/(X_AT_A) = 1/k^2$, we obtain k . Since $(OX_B)/(OX_A) = k$, the reflection event X_B is now determined. Now, the reception event T_B is determined by tracing the reflected light ray back onto the moving worldline. This gives the time dilation effect. The longitudinal mirror is also determined by drawing through X_B the line that is parallel to OH . This gives the length contraction effect.

38, 218 (1969); L. Parker and G.M. Schmieg, “A useful form of the Minkowski diagram,” *Am. J. Phys.* **38**, 1298 (1970); E.N. Glass, “Lorentz boots and Minkowski diagrams,” *Am. J. Phys.* **43** (11), 1013-1014 (1975).

V. Conclusion

[fix]

By drawing the spacetime diagram of a Michelson-Morley apparatus, we have obtained an accurate visualization of the proper-time elapsed along a piecewise-inertial observer’s worldline.

The ideas presented in this paper are being implemented in a series of interactive computer programs which we will eventually be posted to the following websites:

<http://physics.syr.edu/courses/modules/LIGHTCONE/java/>

and

<http://physics.truman.edu/>.

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